Review

Discrete Mathematics, Algorithms and Applications
Vol. 11, No. 5 (2019) 1930002 (18 pages)

DOI: 10.1142/S1793830919300029

A survey on Hamiltonicity in Cayley graphs and digraphs on different groups

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Received 18 March 2019
Revised 25 June 2019
Accepted 8 August 2019
Published 8 October 2019

Lovász had posed a question stating whether every connected, vertex-transitive graph has a Hamilton path in 1969. There is a growing interest in solving this longstanding problem and still it remains widely open. In fact, it was known that only five vertex-transitive graphs exist without a Hamiltonian cycle which do not belong to Cayley graphs. A Cayley graph is the subclass of vertex-transitive graph, and in view of the Lovász conjecture, the attention has focused more toward the Hamiltonicity of Cayley graphs. This survey will describe the current status of the search for Hamiltonian cycles

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and paths in Cayley graphs and digraphs on different groups, and discuss the future direction regarding famous conjecture.

Keywords: Cayley graph and digraph; vertex-transitive graph; Hamiltonian paths and cycles.

Mathematics Subject Classification 2010: 05C45, 68R05

1. Introduction

Precisely, a Hamiltonian path is a path that visits every vertex of a graph exactly once. If there is an edge between the starting and ending point of a Hamiltonian path then it is a Hamiltonian cycle. The general problem of finding the existence of a Hamiltonian cycle is, the well-known Hamiltonicity problem. The problem of finding Hamiltonian cycles in graphs can be studied in all kind of graphs, and this survey deals with the Lovász conjecture about the existence of Hamiltonian paths and cycles in the special case of vertex-transitive graphs called Cayley graphs. In 1878, Arthur Cayley had introduced Cayley graphs to explain the concept of abstract groups which are described by a set of generators.

Definition 1.0.1. The Cayley graph $\mathrm{Cay}(G, S)$ of a group G with respect to the generating set $S(\subset G)$ is defined to be the undirected graph in which the vertex set is given by the elements of G and the edge set is given by $\{\{g, g.s\} \mid g \in G, s \in (S \cup S^{-1})\backslash\{e\}\}$.

Similarly a directed graph is called the Cayley digraph of G with generating set S, $\overrightarrow{\mathrm{Cay}}\,(G; S)$, if each element of G is a vertex of $\overrightarrow{\mathrm{Cay}}\,(G; S)$, and for all x, $y \in G$ there is an arc from x to y if and only if $xs = y$ for some $s \in S$.

The well-known conjecture that every Cayley graph on a finite group of order, at least, three has a Hamiltonian cycle is still open and it was proven to be true for Cayley graphs on certain class of groups. Many attempts utilizing different solving techniques (such as [2, 45]) have been made; it was even proven that almost all Cayley graphs are Hamiltonian [26], but a complete solution is yet to be found.

Most of the results about the conjecture on Cayley graphs were first surveyed in 1984 by Witte and Gallian [50]. Furthermore, in 1996 results on Hamiltonian cycles and paths in Cayley graphs and digraphs were surveyed by Curran and Gallian [15] which provide ample background on previous work.

It is important to determine which groups with which generating sets have Hamiltonian paths and cycles. According to the previous survey results, the problem was solved for all Cayley graphs on abelian groups, but it was not yet proven on all non-abelian groups. Especially, dihedral groups are considered to be important in the search for Hamiltonian cycles in Cayley graphs on finite groups since there has been little progress made to generalize the results on Cayley graphs of dihedral groups. In the case of p-groups, Witte had proven in general that a connected Cayley digraph of any p-group has a Hamiltonian cycle.

Today, the constant stream of results in this area continues to provide with new and interesting theorems and still further questions. Since this area is so vast, in this survey the main results on Cayley graphs and digraphs of important groups will be covered. Therefore, the main objective of the survey is to provide a better picture of the famous conjecture on Hamiltonian paths and cycles in Cayley graphs, as it exists today.

2. Abelian Groups

If Hamiltonicity results of Cayley digraphs on abelian group are considered, it was already proven by Gallian [18] that Cayley digraph on abelian group has Hamiltonian path for any nonempty generating set, using the proof by induction on the number of elements in the generating set. However, some connected Cayley digraphs on abelian group do not have a Hamiltonian cycle. For example, when the Cayley digraph $\overrightarrow{\mathrm{Cay}}\,(\mathbb{Z}_{12}; 3, 4)$ is considered it does not have a Hamiltonian cycle.

Suppose H is a Hamiltonian cycle of the Cayley digraph $\overrightarrow{\mathrm{Cay}}\,(\mathbb{Z}_{12}; 3, 4)$. If a vertex x of the Cayley digraph travels by 4, then the adjacent vertex $x+1$ cannot travel by 3 due to the collision at $x+4$. So, $x+1$ must travel by 4. Therefore, by induction, every vertex travels by 4. Also, if no vertex travels by 4 then it implies all travel by 3. Hence no Hamiltonian cycle exists in Cayley digraph $\overrightarrow{\mathrm{Cay}}\,(\mathbb{Z}_{12}; 3, 4)$.

The "valency" or the "degree" of a vertex of a graph is the number of edges incident to the vertex with loops counted twice. The "maximum/minimum valency" of a graph is the maximum/minimum valency of its vertices. For a regular graph (X), the valency of all the vertices are equal and hence the maximum valency is the same as the minimum valency which can simply be called as the "valency" of the graph $(\mathrm{val}(X))$.

Two main properties that can be present in a Hamiltonian graph are Hamilton-connectedness and Hamilton-laceability. A graph X is Hamilton-connected if every two vertices of X are connected by a Hamiltonian path [11]. A graph X is Hamilton-laceable if it is bipartite and there is a Hamiltonian path between every pair of vertices that are in different partite sets.

In 1981, Chen and Quimpo have proven the following very remarkable theorem regarding the Hamilton-connectedness and Hamilton-laceability of Cayley graphs of abelian groups.

Theorem 2.0.1 ([13]). *Let X be a connected Cayley graph on an abelian group. If* $\mathrm{val}(X) = 2$, *X is a cycle. If* $\mathrm{val}(X) > 2$, *then*

(1) *X is Hamilton-connected if X is not bipartite.*
(2) *X is Hamilton-laceable if X is bipartite.*

In 1985, Alspach [1] has conjectured the existence of k Hamiltonian cycles in any $2k$-regular connected Cayley graph $\mathrm{Cay}(G; S)$ on a finite abelian group G.

Partially answering Alspach's conjecture, Bermond *et al.* in [10] have proven that any 4-regular connected Cayley graph on a finite abelian group can be decomposed into two Hamiltonian cycles, and Westlund has presented Hamilton decomposable 6-regular abelian Cayley graphs in [48].

Moreover, Liu has proven the Theorems 2.0.2 and 2.0.3 for the existence of Hamilton decompositions in $2k$-regular connected Cayley graphs on abelian groups in 1996 and 2003, respectively.

Theorem 2.0.2 ([32]). *If G is an abelian group of odd order and $S = \{s_1, s_2, \ldots, s_k\}$ is a minimal generating set of G, then $\mathrm{Cay}(G; S)$ has a Hamiltonian decomposition.*

Theorem 2.0.3 ([33]). *If G is a finite abelian group of even order at least 4 and $S = \{s_1, s_2, \ldots, s_k\}$ is a strongly minimal generating set of G, then $\mathrm{Cay}(G; S)$ has a Hamiltonian decomposition.*

2.1. *Circulant graphs*

Circulant graphs are Cayley graphs over a cyclic group [9] (mostly circulant graphs will be denoted as C_n or $\mathbb{Z}_n$ or $\mathrm{Circ}(n)$).

From previous survey results, it is well known that every circulant graph has a Hamiltonian cycle and in [24] Heus had proven that if G is a group equal to the product of cyclic groups, then G has at least one Hamiltonian Cayley graph.

Proposition 2.1.1 ([24]). *The cartesian product $C_{n_1} \times \cdots \times C_{n_k}$ of k cycles is Hamiltonian.*

The Cartesian product $C_{n_1} \times \cdots \times C_{n_k}$ of k cycles can be seen as an $n_1 \times n_2 \times \cdots \times n_k$ grid where the vertex $(i_1, \ldots, i_k)$ is adjacent only to the vertices $(i_1 \pm 1 \bmod n_1, i_2, \ldots, i_k)$, $(i_1, i_2 \pm 1 \bmod n_2, \ldots, i_k), \ldots, (i_1, i_2, \ldots, i_k \pm 1 \bmod n_k)$. The proof was done by induction to k.

Although every connected circulant digraphs have been proven to possess a Hamiltonian path, the analogous result is not proven in the case of Hamiltonian cycles. Hamiltonian property of directed circulant graphs can be addressed by the following question.

Question 2.1.2. Do the Cayley digraphs of the additive group $\mathbb{Z}_n$ with respect to any generating set $S(\overrightarrow{\mathrm{Cay}}\,(\mathbb{Z}_n; S))$ have a Hamiltonian cycle?

Proposition 2.1.3 ([38]). *Assume $G = \langle a, b\rangle$ is abelian. Then there is a Hamiltonian cycle in $\overrightarrow{\mathrm{Cay}}\,(G; a, b)$ if and only if there exist $k, l \geq 0$ such that $\langle a^k b^l\rangle = \langle ab^{-1}\rangle$ and $k + l = |G : \langle ab^{-1}\rangle|$.*

Example ([38]). If $\gcd(a, n) > 1$ and $\gcd(a + 1, n) > 1$, then $\overrightarrow{\mathrm{Cay}}\,(\mathbb{Z}_n; a, a + 1)$ does not have a Hamiltonian cycle.

Proposition 2.1.3 can provide several examples of non-Hamiltonian circulant digraphs with generating set having two elements.

Question 2.1.4. Do the circulant graphs with respect to three elements generating set ($\mathrm{Circ}(n; a, b, c)$) have a Hamiltonian cycle?

Locke and Morris in [34] have provided the answer to the above question. They have shown that the following Cayley digraphs are the examples of connected, non-Hamiltonian circulant digraphs $\overrightarrow{\mathrm{Cay}}\,(G; S)$, such that cardinality of generating set $S > 2$ and the identity element of group $e \notin S$.

- $\overrightarrow{\mathrm{Cay}}\,(\mathbb{Z}_{12k}; 6k, 6k+2, 6k+3)$ for any $k \in \mathbb{Z}^+$,
- $\overrightarrow{\mathrm{Cay}}\,(\mathbb{Z}_{2k}; a, b, b+k)$ for $a, b, k \in \mathbb{Z}^+$ iff $\gcd(a, b, k) \neq 1$ or,
  - — $\gcd(a-b, k) = 1$; and
  - — $\gcd(a, 2k) \neq 1$; and
  - — $\gcd(b, k) \neq 1$; and
  - — either a or k is odd; and
  - — a is even, or both b and k are even.

Proposition 2.1.5 by Morris in [38] too provides existence of non-Hamiltonian circulant digraphs with generating set having three elements. Furthermore, [41] highlights the cases where there exists a non-Hamiltonian $(2, 3, c)$-circulant digraph, while proving the existence of a Hamiltonian cycle for the remaining cases.

Proposition 2.1.5 ([38]). *Let G be an abelian group (written additively) and let $a, b, k \in G$, such that k is an element of order 2 (also assume $\{a, b, b+k\}$ consists of three distinct, nontrivial elements of G). If the Cayley digraph $\overrightarrow{\mathrm{Cay}}\,(G; a, b, b+k)$ is connected, but does not have a Hamiltonian cycle, then G is cyclic.*

Morris had proven contrapositive of the result by assuming that G is not cyclic and shown that the Cayley digraph has a Hamiltonian cycle (if it is connected).

Furthermore, the existence of Hamilton decompositions in several $2k$-valent infinite circulant graphs was proven very recently in [12] of 2018. However, the case of circulant digraphs is still open since the existence of Hamiltonian cycle is not yet proven for all generating sets having elements greater than 3.

3. Dihedral Groups

3.1. *Generalized Dihedral groups*

Definition 3.1.1 ([4]). Let H be a finite abelian group. The generalized dihedral group D_H is the group of order $2|H|$ generated by H and τ where $\tau \notin H, \tau^2 = 1$ and $\tau h \tau = h^{-1}$, for all $h \in H$.

The dihedral groups are special cases of generalized dihedral groups D_H when H is a cyclic group.

Definition 3.1.2 ([4]). A family $\mathcal{F}$ of graphs is H^*-connected when every non-bipartite graph in $\mathcal{F}$ is Hamilton-connected and every bipartite graph in $\mathcal{F}$ is Hamilton-laceable.

Dihedral group D_{2n} of order $2n$ contains elements with n rotations $\{t_i : 0 \leq i \leq n-1\}$ and n reflections $\{ft_i : 0 \leq i \leq n-1\}$ which is given by the representation $D_{2n} = \langle t,\ f \,|\, t^n = e,\ f^2 = e,\ ftf = t^{-1}\rangle$. It can be easily shown that $\mathrm{Cay}(D_{2n}, \{f, t\})$ is Hamiltonian, however it has been difficult to prove in general that Cayley graph on D_{2n} with generating set consisting of all the reflections has Hamiltonian cycle. Recently, Alspach *et al.* in [4] have proven the Theorem 3.1.3 on generalized dihedral groups, by showing that every Cayley graph on the dihedral group D_{2n} when n being even has a Hamiltonian cycle.

Theorem 3.1.3 ([4]). *The family of connected Cayley graphs of valency at least* 3 *on generalized dihedral groups, whose orders are divisible by* 4, *is an* H^**-connected family.*

The following are the consequent results obtained from the above theorem.

Corollary 3.1.4 ([4]). *If* X *is a connected Cayley graph on the dihedral group* D_n, n *even, then* X *has a Hamiltonian cycle.*

Corollary 3.1.5 ([4]). *Every edge of a connected Cayley graph on a dihedral group* D_n, n *even, lies in a Hamiltonian cycle.*

In previous survey results it was mentioned that Witte in 1982 has obtained the generalized results of Cayley digraphs on dihedral groups upto the extent of Theorem 3.1.6.

Theorem 3.1.6. *If* $\overrightarrow{\mathrm{Cay}}\ (D_H; S \cap H)$ *is Hamiltonian, then* $\overrightarrow{\mathrm{Cay}}\ (D_H; S)$ *is also Hamiltonian.*

Pastine and Jaume in [44] have generalized the results on Cayley graphs on dihedral groups and proven the following theorem.

Theorem 3.1.7. *Given a generalized dihedral group* D_H *and a generating subset* S, *if* $S \cap H \neq \phi$, *then the Cayley digraph* $\overrightarrow{\mathrm{Cay}}\ (D_H; S)$ *is Hamiltonian.*

It was proven by a recursive algorithm that produces a Hamiltonian circuit in the digraph.

Moreover, in [25] the authors have proven that, every connected Cayley graph on D_{2p}, where p is a prime has a Hamiltonian cycle and in [52], a Hamiltonian decomposition of the Cayley graph on D_{2p} was presented.

4. Special-Order Groups

Most of the research works since 1995, includes the Hamiltonicity results on Cayley digraphs on groups of special order. One of the remarkable results by Witte is that every Cayley digraph of an arbitrary p-group, p a prime, is Hamiltonian. Combining this result with the result proven by Marušič, it can be stated that every connected vertex-transitive graph of order p^k, p a prime and $k \leq 3$, is Hamiltonian. Furthermore, by generalizing the results of Witte and Marušič, Chen had proven that every connected vertex-transitive graph and digraph of order p^4, p a prime, is Hamiltonian. The following is strongest structural result by Keating and Witte.

Theorem 4.0.1 ([28]). *There is a Hamilton cycle in every Cayley graph in a group whose commutator subgroup is cyclic of prime-power order.*

If recent results in [43] are considered, Pak and Radoičić have recently proven the following theorem based on the Cayley graph of a group with special generating set.

Theorem 4.0.2 ([43]). *Every finite group G of size $|G| \geq 3$ has a generating set S of size $|S| \leq \log_2|G|$, such that the corresponding Cayley graph $\mathrm{Cay}(G; S)$ contains a Hamiltonian cycle.*

Kutnar *et al.* in [31] has shown the following theorem for Cayley graphs on groups whose order has few prime factors.

Theorem 4.0.3 ([31]). *Let G be a finite group. Every connected Cayley graph on G has a Hamiltonian cycle if $|G|$ has any of the following forms (where p, q, and r are distinct primes):*

- kp, *where* $1 \leq k < 32$, *with* $k \neq 24$,
- kpq, *where* $1 \leq k \leq 5$,
- pqr,
- kp^2, *where* $1 \leq k \leq 4$,
- kp^3, *where* $1 \leq k \leq 2$.

The comprehensive study of the conjecture in the case of Cayley graphs for which the number of vertices has a prime factorization provides the following result. Every connected Cayley graph on G has a Hamiltonian cycle if

- $|G| = 16p$, where p is prime [16]
- $|G| = 27p, 30p$, where p is prime [19, 20].

Combining above results and Theorem 4.0.3 with the recent result in [42] establishes that, if $|G| = kp$, where p is a prime, with $1 \leq k \leq 47$ then every connected Cayley graph on G has a Hamiltonian cycle. In [42] a computer-assisted proof is presented and the following two independent results were also proven.

Corollary 4.0.4 ([42]). *If $|G| < 144$ (and $|G| > 2$), then every connected Cayley graph on G is Hamiltonian.*

Proposition 4.0.5 ([42]). *If $|G| < 48$, then every connected Cayley graph on G is either Hamiltonian connected or Hamiltonian laceable (or has valence ≤ 2).*

Moreover, the following two theorems proved by Dave Witte Morris state the existence of a Hamiltonian cycle in Cayley graphs of a finite group G, based on the order of its commutator subgroup, $[G, G]$.

Theorem 4.0.6 ([37]). *If G is a nontrivial, finite group of odd order, whose commutator subgroup $[G, G]$ is cyclic of order $p^\mu q^\upsilon$, where p and q are prime, and $\mu, \upsilon \in \mathbb{N}$, then every connected Cayley graph on G has a Hamiltonian cycle.*

Theorem 4.0.7 ([40]). *If the commutator subgroup of G has order $2p$, where p is an odd prime, then every connected Cayley graph on G has a Hamiltonian cycle.*

Corollary 4.0.8 is an immediate consequence of Theorem 4.0.6, which results in a further contribution to Theorem 4.0.3 (as mentioned in Corollary 4.0.9).

Corollary 4.0.8 ([37]). *If G is a nontrivial, finite group of odd order, whose commutator subgroup $[G, G]$ has order pq, where p and q are distinct primes, then every connected Cayley graph on G has a Hamiltonian cycle.*

Corollary 4.0.9 ([37]). *If p and q are distinct primes, then every connected Cayley graph of order $9pq$ has a Hamiltonian cycle.*

When considering the topic of vertex-transitive graphs for which the number of vertices has a prime factorization, Zhang in [51] has obtained the following result for a prime p.

Theorem 4.0.10. *Connected vertex-transitive digraphs of order p^5 are Hamiltonian.*

5. Solvable and Nilpotent Groups

It is well known that, *abelian* $\subset$ *nilpotent* $\subset$ *solvable*. Hamiltonicity of Cayley graphs on abelian group was reviewed, so next it is essential to discuss the results on Cayley graphs on nilpotent and solvable groups.

Question 5.0.1. Does every Cayley graph and digraph on nilpotent groups have a Hamiltonian path or cycle?

It is an open question whether connected Cayley digraphs on nilpotent groups always have Hamiltonian paths. Until 1995, it was known that if G is nilpotent and $|G| = p^n$, where p is a prime, then $\exists$ a Hamiltonian cycle in $\overrightarrow{\text{Cay}}\,(G; S)$.

As mentioned in previous survey results, it was proven that if the commutator subgroup $[G, G]$ of a nontrivial, finite group G is cyclic of prime-power order, then every connected Cayley graph on G has a Hamiltonian cycle. It is natural to try to prove a generalization that only assumes the commutator subgroup is cyclic, without making any restriction on its order, but that seems to be an extremely difficult problem. In 2013, Ghaderpour and Morris [21] replaced the assumption on the order of $[G, G]$ with the rather strong assumption that G is nilpotent and obtained the following result.

Theorem 5.0.2 ([21]). *Let G be a nontrivial, finite group. If G is nilpotent, and the commutator subgroup of G is cyclic, then every connected Cayley graph on G has a Hamiltonian cycle.*

The proof of this Theorem 5.0.2 is based on a variant of the method of Marušič that was established in the proof of Theorem 4.0.1. The above result was obtained by eliminating the restriction on the cardinality of the generating set S given in Theorem 4.0.1, meanwhile Morris had obtained analogous result for the existence of Hamiltonian path in Cayley digraphs on nilpotent groups without making any assumption on the commutator subgroup.

Theorem 5.0.3 ([36]). *Suppose G is a nilpotent, finite group. If $\{a, b\}$ is any 2-element generating set of G, then the corresponding Cayley digraph $\overrightarrow{\text{Cay}}\,(G; a, b)$ has a Hamiltonian path.*

This result implies that all of the connected Cayley graphs of valence ≤ 4 on G have Hamiltonian paths.

Question 5.0.4. Does every Cayley graph and digraph on solvable groups have a Hamiltonian path or cycle?

Milnor's counter example (1975) had shown that it is not true that there is a Hamiltonian path in every Cayley digraph on a solvable group. Morris in [39] has shown that there are infinitely many groups G, such that every Cayley graph on G has a Hamiltonian cycle, and G is not solvable and provided numerous infinite families of finite groups G, for which it is known that every connected Cayley graph on G has a Hamiltonian cycle. However, it seems that the union of these families contains only finitely many groups that are not solvable.

Moreover, [31] presents an unpublished result shown by Kutnar *et al.*, stating that every Cayley graph on A_5 (Alternating group of degree 5) which is the smallest nonsolvable group has a Hamiltonian cycle. Hence, there are infinitely many primes p, such that every Cayley graph on $A_5 \times \mathbb{Z}_p$ has a Hamiltonian cycle. Specifically, Morris (2015) has proven the following result when $p \equiv 1 (\text{mod } 30)$.

Proposition 5.0.5 ([39]). *If p is a prime, such that $p \equiv 1(\text{mod } 30)$, then every connected Cayley graph on the direct product $A_5 \times \mathbb{Z}_p$ has a Hamiltonian cycle.*

6. Direct Product and Semi-Direct Product of Groups

6.1. *Direct product of groups*

Definition 6.1.1 ([35]). Let G_1 and G_2 be two simple graphs with their vertex sets as $V_1 = \{u_1, u_2, \ldots, u_l\}$ and $V_2 = \{v_1, v_2, \ldots, v_m\}$, respectively. Then the direct product of these two graphs denoted by $G_1 \times G_2$ is defined to be a graph with vertex set $V_1 \times V_2$, where $V_1 \times V_2$ is the cartesian product of the sets V_1 and V_2 such that any two distinct vertices (u_1, v_1) and (u_2, v_2) of $G_1 \times G_2$ are adjacent if u_1u_2 is an edge of G_1 and v_1v_2 is an edge of G_2.

The digraph $\text{Cay}(\{(1, 0), (0, 1)\} : \mathbb{Z}_m \times \mathbb{Z}_n)$ is denoted by $\mathbb{Z}_m \times \mathbb{Z}_n$ and it is isomorphic to the cartesian product of a directed m-cycle and a directed n-cycle [15]. The Cartesian product of two Hamiltonian graphs is always Hamiltonian, but the analogous statement is not true for Cayley digraphs. Theorem 6.1.2 on Cayley graph of Cartesian product of directed cycles was proven by Trotter and Erdös (1978) which became one of the first result to expose the necessary conditions for $\mathbb{Z}_m \times \mathbb{Z}_n$ to be Hamiltonian.

Theorem 6.1.2 ([47]). *The Cartesian product $C_{n_1} \times C_{n_2}$ of directed cycles is Hamiltonian if and only if the greatest common divisor (g.c.d.) d of n_1 and n_2 is at least two and there exist positive integers d_1, d_2 so that $d_1 + d_2 = d$ and* $\text{g.c.d.}(n_1, d_1) = \text{g.c.d.}(n_2, d_2) = 1$.

According to [8], the vertex set of the k^{th} cartesian power of a directed cycle of length m can be naturally identified with the abelian group $(\mathbb{Z}_m)^k$. For any two elements $v = (v_1, \ldots, v_k)$ and $w = (w_1, \ldots, w_k)$ of $(\mathbb{Z}_m)^k$, it is easy to see that if there is a Hamiltonian path from v to w, then,

$$v_1 + \cdots + v_k \equiv w_1 + \cdots + w_k + 1 \pmod{m} \tag{1}$$

The authors, Austin *et al.* have proven the converse of the above result in [8], for $k \neq 2$ and m is odd.

As emphasized in previous survey results [15], it was shown by Jungreis *et al.* that every Cayley graph on a group of the forms $\mathbb{Z}_p \times A_4$ (p be prime, A_4 be the alternating group of degree 4) and $D_{2p} \times D_{2p}$ (prime $p \neq 2$ and D_{2p} is the dihedral group of order $2p$) are Hamiltonian in regard to the direct product of Cayley graphs of group of low order. More recently, Andruchuk *et al.* in [6] have developed those results and obtained the following Theorem 6.1.3.

Theorem 6.1.3 ([6]). *A Cayley digraph on $D_{2n} \times D_{2m}$ with outdegree two is Hamiltonian if and only if it is connected.*

In Theorem 6.1.3, for an integer $n \geq 2$, the symbol D_{2n} denotes the dihedral group of order $2n$. For $n \geq 3$ this is the group of symmetries of regular n-gon under the operation of function composition, and it has the presentation $\langle R, F \mid R^n = e, F^2 = e, FRF = R^{-1} \rangle$, where R is the counterclockwise rotation of $360/n^\circ$ and F is a reflection across any axis of symmetry. For $n = 2$ the same presentation can be used to define D_4 and $D_4 \approx \mathbb{Z}_2 \times \mathbb{Z}_2$ [6].

6.2. *Semi-direct product of groups*

If K and L are groups, then a semi-direct product of K by L is a group G such that K is a normal subgroup of G, L is a subgroup of G, $K \cap L$ is the identity element of G and $K \cup L$ generates G.

In previous survey results, it was shown that every connected Cayley graph on a semi-direct product of a cyclic group of prime order by an abelian group is Hamiltonian [17] and sufficient conditions for semi-direct product of two cyclic groups to have a directed Hamiltonian cycle. There seems to be less progress made with regard to semi-direct product of Cayley graphs but at present Morris has shown many Cayley digraphs that do not have a Hamiltonian path and obtained the following result.

Theorem 6.2.1 ([38]). *For any $n \in \mathbb{N}$, there is a connected Cayley digraph* $\overrightarrow{\text{Cay}}$ $(G; a, b)$*, such that*

(1) $\overrightarrow{\text{Cay}}$ $(G; a, b)$ *does not have a Hamiltonian path,*
(2) *a and b both have order greater than n.*

Furthermore, if p is any prime number such that $p > 3$ and $p \equiv 3 (\text{mod } 4)$, then examples of connected Cayley digraphs that do not have Hamiltonian path such that the commutator subgroup of G has order p was constructed by considering $G = \mathbb{Z}_m \ltimes \mathbb{Z}_p$, which is a semi-direct product of two cyclic groups.

7. Coxeter Groups

In 1989, it was shown by Conway *et al.* that if G is a finite group generated by reflections $R_1, \ldots, R_n$ [14], then there is a Hamiltonian circuit in the Cayley diagram for G corresponding to these generators. However, they have proven there is a presentation for every finite Coxeter group so that the corresponding Cayley graph has a Hamilton cycle. In this section, the notion of a Coxeter group of type A_n, B_n and D_n will be briefly introduced, and the important recent results about the existence of Hamilton cycles in Cayley graphs of finite Coxeter groups of type A_n, B_n and D_n will be mainly focused.

7.1. *Symmetric groups (Coxeter Group of type A_n)*

A Coxeter group is a group generated by reflections $R_1, R_2, \ldots, R_n$ such that the only other relations are of the form $(R_i R_j)^k = 1$. Given a Coxeter group G, we associate a graph with it, called a Coxeter diagram, where there is a vertex associated with each of the generating reflections [3]. The symmetric group S_n is the group of all the permutations of $\{1, \ldots, n\}$.

Definition 7.1.1 ([24]). An "involution" is a group element of order two.

Definition 7.1.2 ([24]). A cycle is a permutation f for which there exists an element x in $\{1, \ldots, n\}$ such that $x, f(x), \ldots, f^k(x) = x$ are the only elements moved by $f \cdot k$ is called the length of the cycle and is equal to its order. Cycles of length two are called "transpositions".

Konstantinova has investigated previous results of Cayley graphs on symmetric groups and shown that Cayley graphs on the symmetric group generated by any sets of transpositions are Hamiltonian. Independently, number of results were shown for particular sets of generators based on transpositions and following are the results relevant to Cayley graphs on special sort of symmetric groups [29].

- In 1991, Jwo *et al.* had shown that the star graph is Hamiltonian.

- Jwo had also shown that the bubble-sort graph is Hamiltonian.
- Hamiltonicity of Pancake graph had been investigated by Sheu *et al.* in 1999. The Pancake graph P_n is Hamiltonian for any $n \geq 3$.

The following theorem on Cayley graphs of symmetric group was proven by Pak and Radoičić in 2004.

Theorem 7.1.3 ([24]). *If S_n is generated by three involutions α, β, γ such that two of them commute, then the Cayley graph* $\mathrm{Cay}(S_n, \{\alpha, \beta, \gamma\})$ *is Hamiltonian.*

Heus in [24] has proven Theorem 7.1.3 by induction that there exists a spanning cycle of Hamilton cycle through S_n.

It was already proven that $\mathrm{Cay}(S_n, X)$ is Hamilton-laceable when X is a generating set of transpositions for S_n.

Theorem 7.1.4 ([24]). *The Cayley graph* $\mathrm{Cay}(S_n; S)$ *is Hamiltonian if S consists only of transpositions.*

However, Araki in [7] has provided a strong generalization regarding all of the connected Cayley graphs on the symmetric group, whose connection sets contain only transpositions, are Hamilton-laceable.

In the case of Cayley digraph on symmetric group S_n, $\sigma - \tau$ is an important graph. In 1995 survey, it was shown upto the point that Cayley graph on S_n with generators $\sigma = (12 \ldots n)$ and $\tau = (12)$, $\mathrm{Cay}(S_n; \{\sigma, \tau\})$ is Hamiltonian for all $n > 2$, and recently Williams (2013) in [49] had considered the directed $\sigma - \tau$ graph $\mathcal{G}(n) = \overrightarrow{\mathrm{Cay}}\,(S_n; \{\sigma, \tau\})$ on the symmetric group S_n with generators $\sigma = (12 \ldots n)$ and $\tau = (12)$, and edges $\epsilon_\sigma \cup \epsilon_\tau$ and constructed Hamiltonian path for all n and a Hamiltonian cycle for odd n.

Although the Hamiltonicity of $\mathrm{Cay}(S_n, S)$ has been proved for many S, there are still many more open cases remaining to be solved. It is known from Theorem 7.1.4 that every connected bipartite Cayley graph on the finite Coxeter group of type A_n, $n \geq 2$, whose connection set contains only transpositions and has valency at least three is Hamilton-laceable. In 2014, Alspach [3] has considered finite Coxeter groups A_n, B_n and D_n with regard to the problem of whether they are Hamilton-laceable or Hamilton-connected and obtained analogous results for connected bipartite Cayley graphs on finite Coxeter groups of type B_n and D_n which is given in Theorems 7.2.1 and 7.3.1.

7.2. *Wreath products (Coxeter Group of type B_n)*

Consider the Coxeter diagram shown in Fig. 1.

The generator R_i, $1 \leq i \leq n-1$, is the reflection of E^n through the orthogonal complement of the vector with $\bar{1}$ in coordinate i, 1 in coordinate $i+1$ and zeros in all other coordinates. This is the Coxeter group B_n and it is easy to see that it is isomorphic to the wreath product $S_n \wr S_2$ [3].

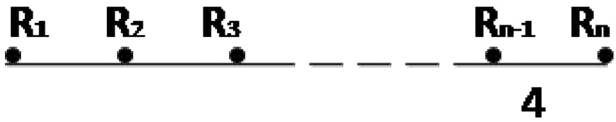

Fig. 1. Coxeter diagram.

Theorem 7.2.1 ([3]). *If* $X = \mathrm{Cay}(S_n \wr S_2; S)$ *is connected, has valency at least three, and* S *contains only double transpositions and transpositions, then* X *is bipartite and Hamilton-laceable.*

7.3. *Coxeter group of type* D_n

A natural subgroup for each of groups with the signed permutations of length n, is the collection of signed permutations with an even number of negative terms. This group is the Coxeter group D_n. It is easy to see that D_n has index 2 in $S_n \wr S_2$ [3].

Theorem 7.3.1 ([3]). *If* $X = \mathrm{Cay}(D_n; S)$ *is a connected Cayley graph of valency at least* 3 *on* D_n, $n \geq 2$, *such that* S *contains only double transpositions and double negators, then* X *is Hamilton-laceable when it is bipartite, or Hamilton-connected when it is not bipartite.*

Meanwhile, the result of Conway *et al.* (1989) has been generalized for real reflection groups by Kriloff and Lay in [30] and consequently they have shown Theorem 7.3.2 which says that there exists Hamiltonian cycles for every Cayley graph of highly non-abelian infinite family of complex reflection groups, $G = G(de, e, n) \cong \mu^n \ltimes S_n$ with respect to commonly used generating sets of reflections.

"We describe an infinite family of complex reflection groups. Let $d, e, n \geq 1$, μ denote the cyclic group of de-th roots of unity, and let ζ generate μ. Under the *standard monomial representation*, $G(de, e, n)$ consists of monomial matrices with nonzero entries $\zeta^{a_1}, \ldots, \zeta^{a_n}$ such that $(\zeta^{a_1}, \ldots, \zeta^{a_n})^d = 1$, or equivalently $a_1 + \cdots + a_n \equiv 0 \bmod e$" [30].

"A reflection group G on V (n-dimensional complex vector space) is *imprimitive* if there is a decomposition $V = V_1 \oplus V_2 \oplus \cdots \oplus V_k$ into proper nonzero subspaces such that G permutes the subspaces. The groups $G(de, e, n)$ with $de, n \geq 2$ are imprimitive in their action on a system of lines orthogonal to the reflecting hyperplanes and the exceptional groups are primitive" [30].

Theorem 7.3.2 ([30]). *If* G *is an irreducible imprimitive complex reflection group and* S *is a standard generating set for* G, *then the* (*undirected right*) *Cayley graph* $\Gamma(G, S)$ *has a Hamiltonian cycle.*

It was proven separately for the cases of $d = 2$ in $G(de, e, 2)$ and $G(de, e, 3)$ without distinguishing $e = 2$. Explicit Hamiltonian cycles were given for the families $G(d, 1, 2)$, $G(2e, e, 2)$, $G(e, e, 3)$ and a Hamiltonian cycle in the graph for

$G(de, e, 2)$ was constructed by applying the flipping process to an explicit Hamiltonian path [30].

8. Miscellaneous

8.1. *Cayley graph on Hamiltonian group*

One of the important family of graphs for which Lovász Conjecture on Cayley graphs was established was the family of Hamiltonian groups. A Hamiltonian group is a non-abelian group in which all subgroups are normal as mentioned in [5].

Alspach and Qin in [5] had investigated Cayley graphs on Hamiltonian groups and proved Theorem 8.1.1 which is analogous to the Chen–Quimpo Theorem 2.0.1.

Theorem 8.1.1 ([5]). *Let $X = \mathrm{Cay}(G; S)$ be a connected Cayley graph on a Hamiltonian group G. If $\mathrm{val}(X) \geq 3$, then X is Hamilton-connected if X is not bipartite or X is Hamilton-laceable if X is bipartite.*

The conjunction product $\mathrm{Cay}(G; S) \cdot \mathrm{Cay}(H; T)$ of two Cayley graphs is the graph $\mathrm{Cay}(G \times H, S \times T)$ [24]. An example of a Cayley graph based on the product of groups is the Cayley graph of a Hamiltonian group. Every Hamiltonian group G is a direct product of the form $G = Q_8 \times B \times A$, where Q_8 is the quaternion group, B is a number of copies of $\mathbb{Z}_2$ and A is an odd order abelian group [24].

Keating has proven the following conjecture on product of Cayley graphs is true for all cases except the case where $\mathrm{Cay}(G, S) = C_2$, which is still open [24].

Conjecture 8.1.2. *If there is a Hamilton cycle in each of $\mathrm{Cay}(G, S)$ and $\mathrm{Cay}(H, T)$, then there is a Hamilton cycle in the conjunction $\mathrm{Cay}(G, S) \cdot \mathrm{Cay}(H, T)$, unless it is not connected.*

8.2. *Cubic Cayley graph*

Given a group G and a generating set S of G, the Cayley graph $\mathrm{Cay}(G, S)$ is cubic iff $|S| = 3$ and $S = \{a,\ b,\ c \mid a^2 = b^2 = c^2 = 1\}$ or $S = \{a,\ b,\ b^{-1} \mid a^2 = b^s = 1\}$.

Next result by Glover and Marušič (2006) is about the Hamiltonicity for cubic Cayley graphs arising from groups that have a $(2, s, 3)$-presentation, that is, for groups $G = \langle a, b | a^2 = 1, b^s = 1,\ (ab)^3 = 1, \ldots \rangle$ generated by an involution a and an element b of order $s \geq 3$ such that their product ab has order 3.

Theorem 8.2.1 ([23]). *Let $s \geq 3$ be an integer and $G = \langle a, b \mid a^2 = 1, b^s = 1, (ab)^3 = 1, \mathrm{etc.} \rangle$ be a group with a $(2, s, 3)$-presentation. Then the Cayley graph $X = \mathrm{Cay}(G, \{a, b, b^{-1}\})$ has a Hamilton cycle when $|G|$ (and thus also s) is congruent to 2 modulo 4, and has a cycle of length $|G| - 2$, and thus necessarily a Hamilton path, when $|G|$ is congruent to 0 modulo 4.*

An important result in 2009 about the existence of Hamilton cycles in cubic Cayley graphs by Glover *et al.* is that if $s \equiv 0 (\mathrm{mod}\ 4)$ or s is odd then the Cayley graph $\mathrm{Cay}(G, S)$ has a Hamiltonian cycle and it has been obtained using the theory of maps [22].

Each cubic Cayley graph has a canonical Cayley map given by an embedding of the Cayley graph $X = \mathrm{Cay}(G, \{a, b, b^{-1}\})$ of the $(2, s, 3)$-presentation of a group $G = \langle a, b \mid a^2 = 1, b^s = 1, (ab)^3 = 1, \text{etc.}\rangle$ in the closed orientable surface of genus $1 + (s - 6)\frac{|G|}{12s}$ with faces $\frac{|G|}{s}$ disjoint s-gons and $\frac{|G|}{3}$ hexagons [22].

The proof is done by finding a tree of faces in this canonical Cayley map whose boundary encompasses all vertices of the graph. Essential ingredient in the Hamiltonian tree of faces method is the concept of cyclic edge-connectivity [22].

8.3. *Alternating group graph*

The alternating group graph AG_n is the undirected Cayley graph of the set of $2(n-2)$ generators of the alternating group A_n given by $g_3^-, g_3^+, g_4^-, g_4^+, \ldots, g_n^-, g_n^+$, where $g_i^- = (1, i, 2)$, $g_i^+ = (1, 2, i)$ in permutation cycle notation. It was shown by Jwo *et al.* in 1993 [27] AG_n is Hamiltonian, and in 2012 Su *et al.* [46] have shown that any alternating group graph AG_n, where $n \geq 3$ is an integer, contains $2n - 4$ mutually independent Hamiltonian cycles.

9. Concluding Remarks

In this survey, the following two problems have been studied with the purpose of updating the results on the existence of Hamiltonian paths and cycles in Cayley graphs and digraphs.

(1) Cayley graph/digraph on particular group with which generating set has Hamiltonian paths or cycles.
(2) In general, Cayley graphs/digraphs on which groups have Hamiltonian paths or cycles for all generating sets.

The problem of finding Hamiltonian cycles appeared to be very broad and in this survey the problem was investigated with all known results since 1995 about Hamiltonicity concerning the Cayley graphs on dihedral groups, permutation groups, p-group, coxeter groups, direct product of groups, semi-direct product of groups, nilpotent groups and solvable groups but still there are a lot of open problems remaining unsolved.

On the whole, the conjecture has been true for groups of small order as well as $\mathrm{Cay}(G; S)$ has a Hamiltonian cycle in the cases when G is abelian, when $[G, G]$ is cyclic of prime-power order, when G is of prime-power order, when the order of G is multiples of primes and when G is nilpotent and $[G, G]$ is cyclic [30]. It is important to note that above result is true for all generating sets S of G. These results also signify that many Cayley digraphs do not have Hamiltonian cycles. Especially, it has to be solved that Cayley graphs on dihedral group D_{2n} with the generating set consisting all reflection elements has a Hamiltonian cycle. In 2015, Morris has shown that there are infinitely many nonsolvable groups G, such that every Cayley graph on G has a Hamiltonian cycle. This result validates the contrapositive of the statement that Cayley graphs on solvable groups always have Hamiltonian cycles. Also, it is known that connected Cayley digraphs on solvable groups do not always have Hamiltonian paths, but it is not yet proven that connected Cayley digraphs on nilpotent groups always have Hamiltonian paths.

This survey has reviewed several recent results that were spread around the literature and rare positive results on weaker Lovász conjecture for finite groups that left unanswered in previous survey results, but still there are some current progress in regard to the conjecture.

Acknowledgments

We are grateful to all the authors in the reference because this survey has included most of the Hamiltonicity results on Cayley graphs and digraphs in several research papers which have appeared since the 1995 survey of Witte and Gallian. Many thanks are owed to Prof. Morris, University of Lethbridge whose papers formed the foundation of many of recent results of the survey and some ideas about the current progress in the conjecture. Also, we would like to thank Mr. Sarath Kumara, University of Sri Jayewardenepura who has sincerely contributed in completing this survey.

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