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A new method of generating pythagorean triples

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Abstract

Pythagorean numbers have captured the imagination of the man for their beauty from the times of Pythagoras (i.e., from 582 BC) or possibly before. The most familiar Pythagorean triples are (3,4,5) and (5,12,13). These numbers can be generated[1] algebraically. The purpose of this paper is to introduce a novel method of generating Pythagorean triples.

Key Words: Pythagorean Triples, Generation.

1. Introduction

These numbers are the lengths of the sides of a right triangle. That is

$$3^2 + 4^2 = 5^2 \text{and} 5^2 + 12^2 = 13^2$$

Throughout the history various algebraic formulations[1] have been used to generate these numbers. Pythagoras himself is credited with using the set of formulae

$$X = 2n + 1, Y = 2n^2 + 2n, \text{ and } Z = 2n^2 + 2n + 1$$

to generate these triples. However, this formulation does not account for all right triangles with integral sides. Another widely used set of formulae[1] are

$$X = 2UT, Y = U^2 — T^2, \text{ and } Z = U^2 + T^2.$$

Description

For Pythagorean triples of positive integers X,Y,Z to exist

$$X^2+Y^2-Z^2 = (X-1)X+(Y-1)Y-(Z-1)Z+X+Y-Z=0$$

However, for any integer X; (X-1) X is divisible by the number 2 (i.e., (X-1) X is even).

Hence, (X-1) X + (Y-1) Y - (Z-1) Z is divisible by the number 2.

Therefore, for triples X,Y, Z to exist

$$X^2 + Y^2 - Z^2 = 0 \quad (1)$$

and

$$X + Y - Z = 2S \quad (2)$$

where S is a positive integer, as X +Y> Z.

From equation (2)

$$X^2+2XY+Y^2= Z^2+4ZS+4S^2$$

Combining with equation (1),

$$2XY = 4ZS+4S^2$$

i.e.,

$$Z = \frac{XY}{2S} - S$$

Hence, for Z to be an integer

$$XY = 2PS \quad (3)$$

where P is a positive integer. Then from equation (3)

$$Z = P-S \quad (4)$$

and

$$Y = \frac{2PS}{X}$$

substituting $Y = \frac{2PS}{X}$ and Z = P - S in equation (2), and manipulating, you get

$$X^2 - (P+S)X + 2PS = 0. \quad (5)$$

Similar equation holds for Y. Hence, equation (5) gives X,Y and

$$X,Y = \frac{P+S \pm D}{2} \quad (6)$$

$$\text{Where } D = \sqrt{(P-3S)^2 - 8S^2}$$

But, X and Y are Integers. Hence, a necessary condition for a solution to exist is that D must be an integer.

Hence, for solutions to exist,

$$D^2 = (P-L)^2 = P^2 - 2PL + L^2 = P^2 - 6PS + S^2 \tag{7}$$

where L is an integer.

From equations (6) and (7)

$$X,Y = \frac{P+S \pm (P-L)}{2} \tag{8}$$

hence, $X = \frac{P+(S-L)}{2}$ with L>3S (i.e., Z>X), (9)

and $Y = \frac{S+L}{2}$ where L+S is even (10)

Rearranging the terms in equation (7)

$$P = \frac{(L-S)(L+S)}{2(L-3S)} \quad \text{with } (L+S) \text{ even} \tag{11}$$

Thus, equation (11) gives the relationship between the integers P, L, and S.

In order to obtain the **Integer P,** which is necessary to generate a Pythagorean triple (X,Y,Z), the values of L and S are chosen in equation (11), **with the conditions L>3S and (L+S) being even.**

Generation of Pythagorean triples by this manner is illustrated below.

For example, in the case $L = \frac{9S}{2}$

P = 77S/12 from equation (11),
and taking S=12, then Z = 77—12 = 65, X= 56S/12=56, and Y= 11S/4=33 from equations (4), (9), and (10).
i.e., (Z,X,Y)=(65, 56, 33) with (P,L,S)=(77, 54, 12)

Thus, for each Pythagorean triple (X,Y,Z) there is a complementary triple of Integers (P,L,S).

Similarly, when L=3S+4,
P= (S+2)(S+1) from equation (11),
and taking S=1, then Z= 6-1 =5, X=(S+2)S=3, Y=2S+2=4.
i.e., (Z,X,Y)=(5,3 ,4) with (P,L,S)=(6,7, 1).
Here, L is deliberately chosen such that L>P so that Y becomes greater than X.

When conditions specified with equation (11) are satisfied, integers (Z,X,Y) generated are the Pythagorean triples. However, when these conditions are violated, fractions are generated for Z, X, and Y. These fractions when multiplied by the least common multiple of their denominators (of the three fractions), it produces the three corresponding integers which constitute a Pythagorean triple.

For example, when L=6 and S=1 then P=35/6 giving

$(Z,X,Y) = \left(\frac{29}{6}, \frac{20}{6}, \frac{7}{2}\right)$. Hence, by multiplying by the number six (i.e., to make P an integer), (Z,X,Y) = (29,20,2 1) with "new" (P,L,S)=(35,36,6).

Similarly, it also works for negative numbers as illustrated below. When L=5, S=3, then P=-2, (i.e., $S<L<3S$)
giving (Z,X,Y) = (-5,-3,4).

Hence, equations (4), (9), (10), and (11) can be used to generate Pythagorean triples.

The numbers generated in this manner can compared with that generated by the familiar method $X = 2UT$, $Y = U^2 - T^2$, $Z = U^2 + T^2$. Tables (1) and (2) give such comparison of Pythagorean triples for the first few values of T and U, respectively.

2. Acknowledgment

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3. Reference

1. D.M. Burton, Elementary Number Theory, Published by Mc Graw-Hill (1998), New York.

L	S	P	X, Y, Z	Pythagorean triple	multipication factor	Corresponding values of	
						U	T
4	1	15/2	6,5/2,13/2	12,5,13	2	3	2
5	1	6	4,3,5	4,3,5	1	2	1
6	1	35/6	$\frac{20}{6}, \frac{21}{6}, \frac{29}{6}$	20,21,29	6	5	2
7	1	6	3,4,5	3,4,5	1	2	1
7	2	45/2	$\frac{40}{2}, \frac{9}{2}, \frac{41}{2}$	40,9,41	2	5	4
8	1	63/10	28/10,45/10,53/10	28,45,53	10	7	2
8	2	15	12,5,13	12,5,13	1	3	2
9	1	20/3	8/3,15/3,17/3	8,15,17	3	4	1
9	2	77/6	56/6,33/6,65/6	56,33,65	3	7	4
10	1	99/14	36/14,77/14,85/14	36,77,85	14	9	2
10	2	12	8,6,10	4,3,5	1/2	2	1
10	3	91/2	84/2,13/2,85/2	84,13,85	2	7	6
11	1	15/2	5/2,12/2,13/2	5,12,13	2	3	2
11	2	117/10	72/10,65/10,97/10	72,65,97	10	9	4
11	3	28	24,7,25	24,7,25	1	4	3
12	1	143/18	44/18,117/18,125/18	44,117,125	18	11	2
12	2	35/3	20/3,21/3,29/3	20,21,29	3	5	2
12	3	45/2	36/2,15/2,39/2	12,5,13	2/3	3	2

Table 1. Give the comparison of Pythagorean triples generated for the first few values of L and S with the corresponding values of U and T. Note that some of the X, Y, and Z values are multiplied by a common factor to obtain the triples.

U	T	Pythagorean triples X,Y,Z	Correspomding values of		
			S	L	P
2	1	4,3,5	1	5	6
3	2	12,5,13	2	8	15
4	1	8,15,17	3	27	20
4	3	24,7,25	3	11	28
5	2	20,21,29	6	36	35
5	4	40,9,41	4	14	45
6	1	12,35,37	5	19	42
6	5	60,11,61	5	17	66
7	2	28,45,53	10	80	63
7	4	56,33,65	12	54	77
7	6	84,13,85	6	20	91

Table 2. Give the comparison of Pythagorean triples generated for the first few non-trivial values of U and T with the corresponding values of S and L.