Hamiltonian Cycles in Cayley Graphs of Semidirect Products of Finite Groups

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Abstract. It has been conjectured that every connected Cayley graph of order greater than 2 has a Hamilton cycle. In this paper, we prove that the Cayley graph of $(\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes \mathbb{Z}_q$ with respect to a generating set S, $Cay((\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes \mathbb{Z}_q, S)$, where $S = \{s, t\}$ with $|s| = p$ and $|t| = q$ is Hamiltonian for $p, q > 3$. Furthermore, the existence of a Hamilton cycle in the Cayley graph of a semidirect product of finite groups is proved by placing restrictions on the generating sets. Consequently, the existence of a Hamilton cycle in the Cayley graphs of several isomorphism types of groups of orders $p^n q, p^2 q^2$ and $p^2 qr$, where $n \geq 2$ is also proved.

Key words: Cayley graph, connected and bridgeless, finite groups, Hamilton cycle, perfect matching, semidirect product, standard generating set.

Introduction

All graphs considered in this paper are finite, undirected graphs without loops or multiple edges. For a graph X we let, $V(X), E(X)$ to denote the set of vertices and the set of edges of X, respectively.

Let G be a finite group and S be a subset of G. We define the Cayley graph of G with respect to S as follows, provided that $1 \notin S$ and S is inverse closed.

Definition 1. The Cayley graph of G with respect to S, $Cay(G, S)$ is the graph whose vertices are the elements of G and g is adjacent to gs for all $g \in G, s \in S$.

For a group G which is a direct or a semidirect product, if a generating set for a Cayley graph of each subgroup that is a factor of the product is contained in S, we say that S is a *standard generating set*.

Cayley graphs are a type of vertex-transitive graphs. A graph X is vertex-transitive if for any vertices $x, y \in V(X)$, there exists an automorphism of X which maps x to y. i.e. the automorphism group of X , $Aut(X)$ acts transitively on $V(X)$.

In 1969, L*á*szl*ó* Lov*á*z questioned whether every finite connected vertex-transitive graph consists of a Hamiltonian path. Inspired by this, many studies have been conducted searching for Hamiltonian paths and cycles in vertex-transitive graphs. It was found four non-trivial vertex-transitive graphs exist without a Hamiltonian cycle, namely the Petersen graph, the Coxeter graph and the two graphs derived by replacing each vertex of the Petersen and the Coxeter graphs by a triangle, up to now. However, none of these is a Cayley graph which resulted in a conjecture stating that every connected Cayley graph of order greater than 2 has a Hamiltonian cycle. Refer to the surveys (Curran & Gallian, 1996), (Witte & Gallian, 1984) and (Lanel et al., 2019) for more information regarding studies conducted based on this conjecture. Our study is focused on determining the Hamiltonian property of Cayley graphs of finite groups whose orders have few prime factors.

The following Theorem including the results in (Curran et al., 2012), (Ghaderpour & Morris, 2011), (Ghaderpour & Morris, 2012), (Kutnar et al., 2011), (Morris, 2014), and

(Morris & Wilk, 2018), summarizes the group orders with few prime factors, for which every connected Cayley graph is known to be Hamiltonian.

Theorem 1.

Let G be a finite group. Every connected Cayley graph on G has a Hamiltonian cycle if $|G|$ has any of the following forms, where p, q and r are distinct primes:

i. kp, *where* $1 \leq k \leq 47$,
ii. kpq, *where* $1 \leq k \leq 5$ *and* $k = 9$,
iii. pqr,
iv. kp^2, *where* $1 \leq k \leq 4$,
v. kp^3, *where* $1 \leq k \leq 2$.

The determination of the existence of a Hamiltonian cycle in Cayley graphs of finite groups based on the nature of its commutator subgroups such as in (Keating & Witte, 1985) when the commutator subgroup is a cyclic p-group, in (Morris, 2017) when the commutator subgroup has order $2p$, in (Morris, 2014) when the commutator subgroup of an odd ordered Cayley graph has order pq and in (Ghaderpour & Morris, 2011) when the commutator subgroup of a Cayley graph on a nilpotent group is cyclic, marks a significant contribution in the related results.

In this paper we prove the following results, which further contribute to the above Theorem.

1. In section 3, we prove that $Cay((\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes \mathbb{Z}_q, S)$, such that $S = \{s, t\}$ with $|s| = p$, $|t| = q$ and $p, q > 3$ is Hamiltonian.
2. In section 4,

i. The existence of a Hamiltonian cycle in the Cayley graph of a semidirect product $H \rtimes K$, where H and K are finite groups, $Cay(H, S_1)$ and $Cay(K, S_2)$ are Hamilton-connected and Hamiltonian respectively, with respect to a generating set S such that $S_1 \subseteq S$ and $S_2 \subseteq S$ (standard generating set) is proved.

ii. Using the above result and the existing literature, the Hamiltonian property of several isomorphism types of groups of orders $p^n q, p^2 q^2$ and $p^2 qr$ is proved while placing some restrictions on the generating sets.

iii. The existence of a Hamiltonian cycle in the Cayley graph of $((\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes \mathbb{Z}_p) \rtimes \mathbb{Z}_q$ with respect to a standard generating set, is proved by giving an explicit construction of a Hamiltonian cycle in the Cayley graph.

Moreover, in Proposition 8 at the beginning of section 3, we prove that there exists a perfect matching in the Cayley graph $X - v$ for a vertex $v \in V(X)$, such that the removal of the edges corresponding to the perfect matching results in a connected and bridgeless graph, which is useful in proving the result 1 mentioned above.

Preliminaries

Some definitions and results useful for the proofs given in sections 3 and 4:

A Hamiltonian cycle in a spanning subgraph is also a Hamiltonian cycle in the ambient graph. Therefore, it is sufficient to prove the results for Cayley graphs with respect to irredundant generating sets.

Definition 2. An irredundant generating set for a Cayley graph X is a generating set S such that no proper subset of S generates X.

The *valency* or the *degree* of a vertex of a graph is the number of edges incident to the vertex with loops counted twice.

The *maximum/minimum valency* of a graph is the maximum/minimum valency of its vertices. For a regular graph, the valency of all the vertices are equal and hence the maximum valency is same as the minimum valency which can simply be called the *valency* of the graph. A regular graph of valency 3 is known as a *cubic* graph (Gross et al., 2013).

A subgraph of a graph X is a graph whose vertex set is a subset of $V(X)$ and the set of edges is a subset of $E(X)$. A subgraph of a graph X, whose vertex set is a subset of $V(X)$ and edges set is a subset of $E(X)$ consisting of all edges connecting pairs of vertices in that subset of vertices is known as an *induced subgraph*. A subgraph consisting of all the vertices of a graph is known as a *spanning subgraph*.

The decomposition or the partitioning of the edges of a graph in to cycles is identified as a *cycle decomposition* in a graph.

A graph X is called *connected* if it is non-empty and any two of its vertices are linked by a path in X. A connected graph is called 2-*connected*, if for every vertex $v \in V(X)$, $X - v$ is connected.

A *bridge* of a connected graph is an edge of the graph whose removal disconnects the graph. A *bridgeless graph* is a graph with no bridges.

In a graph X, for any $u, v \in V(X)$, two paths from u to v is said to be *internally disjoint* if there is no common vertex belonging to both paths. i.e. if the two paths have no common internal vertex.

The following Theorem is useful in determining the connectedness and bridgelessness of graphs in the proofs of our main results in section 3.

Theorem 2. (Zhao, 2011)

A graph X of order $n \geq 3$ is 2-connected iff any two vertices of X are connected by at least two internally disjoint paths.

When there are at least two internally disjoint paths between any two vertices of a graph, the graph is bridgeless as well.

A *matching* or an *independent edge set* of a graph is a set of edges where no two edges share a common vertex. A *perfect matching* or a *1-factor* of a graph is a matching where every vertex of the graph is incident with exactly one edge of the matching.

Every connected vertex-transitive graph on an even number of vertices has a perfect matching, and each vertex in a connected vertex-transitive graph on an odd number of vertices is missed by a matching that covers all remaining vertices (Godsil & Royle, 2001). i.e. in a connected vertex-transitive graph X, if X has even order, then X has a perfect matching and if X has odd order, $X - v$ has a perfect matching for every $v \in X$.

The existence of a perfect matching in a vertex-transitive graph was the main motivation for the presentation of a perfect matching in the Cayley graph, in the proof of Proposition 8. We have also employed the perfect matchings in bridgeless, cubic graphs in the proof of Theorem 9.

Theorem 3. (Petersen, 1891)

For every bridgeless cubic graph, there is a 1-factor containing any specific edge.

A *bipartite graph (bigraph)* is a graph whose vertices can be decomposed in to two disjoint sets such that no two vertices within the same set are adjacent.

Theorem 4. (König, 1936)
A graph is bipartite if and only if it contains no odd cycle.

A graph X is *Hamilton-connected* if there exists a Hamiltonian path between every two vertices of X. A connected bipartite graph X is *Hamilton-laceable* if there exists a Hamiltonian path between all pairs of vertices u and v, where u belongs to one set of the bipartition and v to the other. Chen and Quimpo (1981) had proved the following Theorem related to the Hamilton-connected and Hamilton-laceable properties of the Cayley graphs of abelian groups which is utilized in the proofs of our results in section 4.

Theorem 5. (Chen & Quimpo, 1981)
Let X be a connected Cayley graph on an abelian group. If $val(X) = 2$, X is a cycle. If $val(X) > 2$, then

i. *X is Hamilton-connected if X is not bipartite.*
ii. *X is Hamilton-laceable if X is bipartite.*

We recall the Theorem 6 and 7 proved by the authors of (Morris, 2014) and (Keating & Witte, 1985) since they are useful in determining the Hamiltonian property of Cayley graphs whose orders have few prime factors based on the properties of the commutator subgroups.

Theorem 6. (Morris, 2014)
Odd order Cayley graphs with commutator subgroup of order pq are hamiltonian.

Theorem 7. (Keating & Witte, 1985)
If the commutator subgroup $[G, G]$ of G is a cyclic p-group, then every connected Cayley graph on G has a hamiltonian cycle.

Throughout this paper, let p, q and r to be distinct primes.

An analysis of the Cayley graph of $(\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes \mathbb{Z}_q$ with respect to a generating set $S = \{s, t\}$, where $|s| = p, |t| = q$ and $p, q > 2$:

The Cayley graph of $\mathbb{Z}_p \times \mathbb{Z}_p$ with respect to a generating set $S' = \{a, b\}$, where $|a| = |b| = p$, resembles a torus which is also referred as a $p \times p$ grid structure in some contexts. The torus structure is composed with p-cycles generated due to the two elements a and b.

When considering the Cayley graph, $X'' = Cay((\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes \mathbb{Z}_q, S'')$, where $S'' = \{a, b, c\}$, with $|a| = |b| = p, |c| = q$, the vertices of the graph can be arranged in such a manner that q copies of tori connected along a q-cycle are distinctly identifiable as shown in Figure 1. The q copies of tori represent the subgroup $\mathbb{Z}_p \times \mathbb{Z}_p$ and its cosets in X''.

Maintaining the vertices of X'' in the same arrangement, erase the edges and now draw the Cayley graph $X = Cay((\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes \mathbb{Z}_q, S)$, where $S = \{s, t\}$, with $|s| = p, |t| = q$. Note that the Cayley graph X is 4-regular. The induced subgraphs over the vertices representing the subgroup $\mathbb{Z}_p \times \mathbb{Z}_p$ and its cosets in X consists p number of p-cycles generated due to s.

The p number of p-cycles of $\mathbb{Z}_p \times \mathbb{Z}_p$ and its cosets are connected along a q-cycle. A rough sketch of this graph is shown by Figure 2.

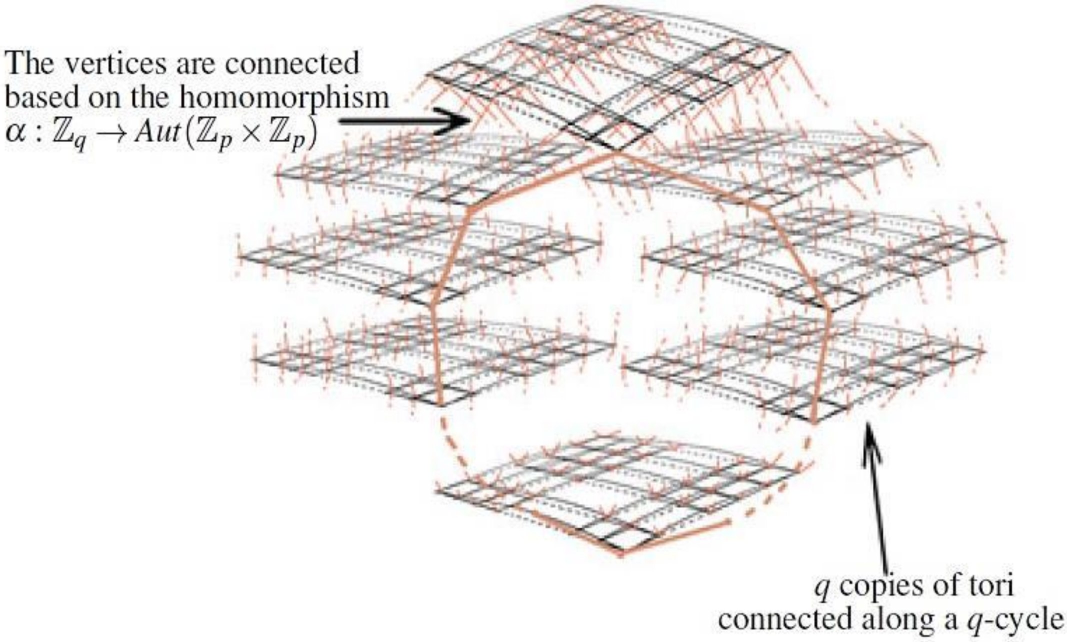

Figure 1: An illustration of the arrangement of subgraphs relevant to the cosets of $(\mathbb{Z}_p \times \mathbb{Z}_p)$ and q-cycles in X''

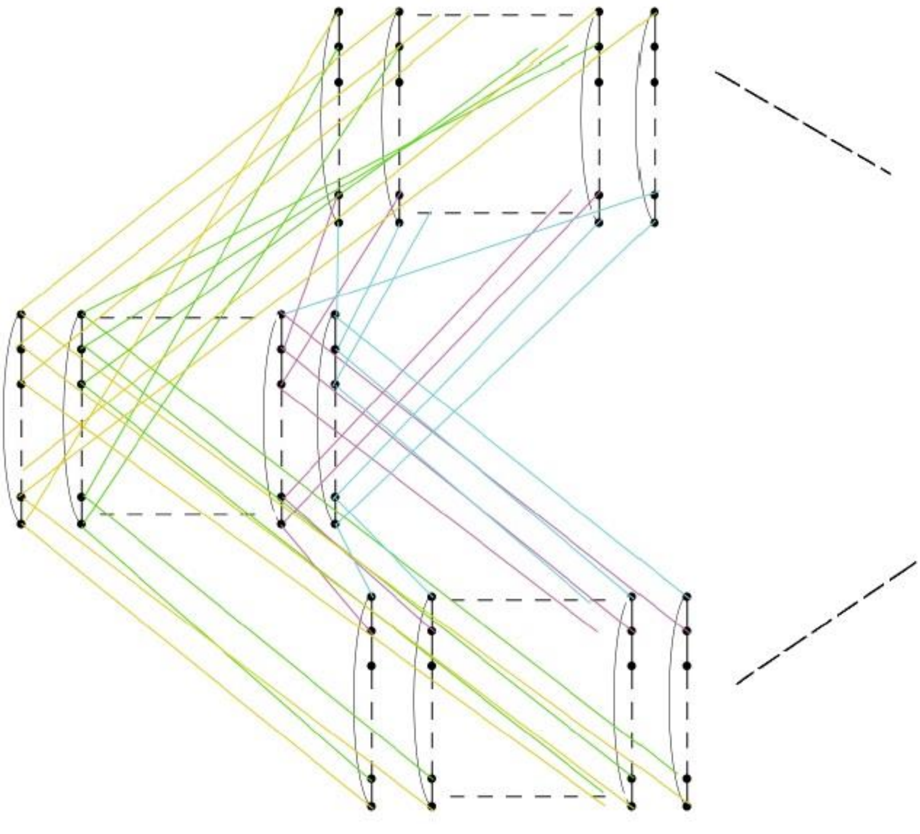

Figure 2: Sketch of X illustrating p number of p-cycles corresponding to cosets of $(\mathbb{Z}_p \times \mathbb{Z}_p)$ in $(\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes \mathbb{Z}_q$

Between two induced subgraphs say, H and I, over the vertices corresponding to two cosets of $\mathbb{Z}_p \times \mathbb{Z}_p$, placed adjacent to each other along the q-cycle, there exists edges from a p-cycle in H to each of the p-cycles in I and vice versa. Considering the p-cycles belonging to a subgraph H, identify adjacent p-cycles such that edges from two adjacent p-cycles in H to a p-cycle in I are incident on adjacent vertices in that p-cycle in I and vice versa.

Arrange (or identify) p-cycles in any subgraph H in an order, using the scripts 0 to $p-1$ as described below, so that the consecutive numbers are assigned to p-cycles such that edges from two adjacent p-cycles (where 0^{th} and $(p-1)^{th}$ p-cycles will also behave adjacent maintaining a cyclical pattern) are incident on two adjacent vertices on a p-cycle in any other subgraph which is placed adjacent to H, when considering the arrangement of subgraphs along the q-cycle as mentioned above.

Let the vertices representing elements of a subgraph H be denoted by $h_{i\ mod\ p, j\ mod\ p}$. The notation $h_{i,j}$ is used to denote $h_{i\ mod\ p, j\ mod\ p}$ throughout this paper and all the computations and representations are considered under $mod\ p$ or $mod\ q$.

The vertices of a subgraph H are connected to the vertices of another such subgraph upon multiplication by t (or t^{-1}). Let the mapping of $v \to v \cdot t$, for any vertex $v \in X$ be represented by bijective functions $f_0, f_1, \cdots, f_{p-1}$, starting with the induced subgraph over the vertices corresponding to the subgroup $\mathbb{Z}_p \times \mathbb{Z}_p$. i.e. if the induced subgraph representing the subgroup $(\mathbb{Z}_p \times \mathbb{Z}_p)$ is H_0, and the induced subgraph over the vertices obtained by multiplying each vertex of H_0 by t is H_1, and the induced subgraph over the vertices obtained by multiplying each vertex of H_1 by t is H_2, etc., then, $f_0 : V(H_0) \to V(H_1)$, $f_1 : V(H_1) \to V(H_2)$, ..., $f_{p-1} : V(H_{p-1}) \to V(H_0)$, where $f_i(v_i) = v_i \cdot t$, for all $v_i \in H_i, 0 \leq i \leq p-1$.

Let directed edges exist from H to I via f_k, where $k \in \{0, 1, \dots, p-1\}$. Let us adapt the following notation for vertices in the subgraph H (with respect to H, I and f_k. i.e. (H, I, f_k) triplet).

Let the coset representative of H be $h_{0,0}$ and the p-cycle containing $h_{0,0}$ be 0^{th} p-cycle of H. Name the remaining vertices in the 0^{th} p-cycle by $h_{0,1}, h_{0,2}, \cdots, h_{0,p-1}$ such that $h_{0,i} = h_{0,0} \cdot s^i, 0 \leq i \leq p-1$. Let the p-cycle including the coset representative $i_{0,0}$ be 0^{th} p-cycle of I and the p-cycle including $f_k(h_{0,1})$ be the 1^{st} p-cycle of I.

The vertices $f_k(h_{0,1}) \cdot s$ and $f_k(h_{0,1}) \cdot s^{-1}$ in I are incident to by vertices from two p-cycles in H and let those p-cycles be identified as adjacent p-cycles to the 0^{th} p-cycle of H. Let the p-cycle including the vertex incident on $f_k(h_{0,1}) \cdot s$ be called the 1^{st} p-cycle of H and the p-cycle including the vertex incident on $f_k(h_{0,1}) \cdot s^{-1}$ the $(p-1)^{th}$ p-cycle of H. Identify the vertex $h_{1,0}$ in the 1^{st} p-cycle of H such that $f_k(h_{1,0}) = f_k(h_{0,1}) \cdot s$. Name the remaining vertices in the 1^{st} p-cycle by $h_{1,1}, h_{1,2}, \cdots, h_{1,p-1}$ such that $h_{1,i} = h_{1,0} \cdot s^i, 0 \leq i \leq p-1$.

Similarly identify the vertices $h_{2,0}, h_{3,0}, \cdots, h_{p-1,0}$, such that $f_k(h_{2,0}) = f_k(h_{0,1}) \cdot s^2$, $f_k(h_{3,0}) = f_k(h_{0,1}) \cdot s^3, \cdots$, $f_k(h_{p-1,0}) = f_k(h_{0,1}) \cdot s^{p-1}$ in 2^{nd} to $(p-1)^{th}$ p-cycles of H and name the remaining vertices in each p-cycle following the same notational convention. Let the p-cycle including $f_k(h_{1,1}), f_k(h_{2,2}), \cdots, f_k(h_{p-1,p-1})$ be the $1^{st}, 2^{nd}, \cdots, (p-1)^{th}$ p-cycles of I, when (H, I, f_k) triplet is considered.

The vertices of I can be named similarly, with respect to f_{k+1} and the next adjacent subgraph (to I). There, let the p-cycle consisting of the coset representative of I be the 0^{th} p-cycle of I as mentioned above but rename the remaining p-cycles and vertices, now considering f_{k+1} and the next adjacent subgraph.

Justification:

Let,

i. $\theta: \mathbb{Z}_q \to Aut(\mathbb{Z}_p \times \mathbb{Z}_p)$ *be the homomorphism determining the semidirect product* $(\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes \mathbb{Z}_q$,

ii. $\theta(i)$ *be denoted as* θ_i *for* $0 \leq i \leq q-1$,

iii. $h_{m,n} = (h,k);\ h \in (\mathbb{Z}_p \times \mathbb{Z}_p),\ k \in \mathbb{Z}_q,\ h_{m,n} \in H$,

iv. $h_{m,n} \cdot s = (h \cdot s', k),\ h_{m,n} \cdot s^2 = (h \cdot s'^2, k),\ \cdots,\ h_{m,n} \cdot s^{p-1} = (h \cdot s'^{p-1}, k);\ s' \in (\mathbb{Z}_p \times \mathbb{Z}_p)$

v. *for any* $(h,k) \in H,\ (h \cdot s'', k) = (s'' \cdot h, k);\ s'' \in (\mathbb{Z}_p \times \mathbb{Z}_p)$,*since* $(\mathbb{Z}_p \times \mathbb{Z}_p)$ *is an abelian group,*

vi. $t = (h', k');\ h' \in (\mathbb{Z}_p \times \mathbb{Z}_p), k' \in \mathbb{Z}_q$,

vii. $f_k(h_{m,n}) = (h,k) \cdot (h', k') = (\theta_{k'}(h) \cdot h', k \cdot k') = (i,j)$, *where* $i \in (\mathbb{Z}_p \times \mathbb{Z}_p), j \in \mathbb{Z}_q, (i,j) \in I$,

viii. $(i,j) \cdot u = (i \cdot u', j)$, *for an element* $u (\neq s)$ *of order* p; $u' \in (\mathbb{Z}_p \times \mathbb{Z}_p)$ *(if the generating set* S *consisted of another element of order* p, *tori will be apparent as subgraphs in the Cayley graph* X. *Starting with a vertex belonging to an induced subgraph over the vertices corresponding to a coset of* $(\mathbb{Z}_p \times \mathbb{Z}_p)$, *the remaining vertices corresponding to the coset can be traversed by multiplication by* s *and another element of order* p*).*

If $f_k(h_{m,n} \cdot s) = (h \cdot s', k) \cdot (h', k') = (s' \cdot h, k) \cdot (h', k') = (\theta_{k'}(s' \cdot h) \cdot h', k \cdot k') = (i,j) \cdot u = (i \cdot u', j) = (u' \cdot i, j)$, *then*

$$f_k(h_{m,n} \cdot s^2) = (h \cdot s'^2, k) \cdot (h', k') = (s'^2 \cdot h, k) \cdot (h', k') = (\theta_{k'}(s'^2 \cdot h) \cdot h', k \cdot k')$$
$$= (u'^2 \cdot i, j) = (i,j) \cdot u^2$$

$f_k(h_{m,n} \cdot s^3) = (h \cdot s'^3, k) \cdot (h', k') = (s'^3 \cdot h, k) \cdot (h', k') = (\theta_{k'}(s'^3 \cdot h) \cdot h', k \cdot k') = (u'^3 \cdot i, j) = (i,j) \cdot u^3$ *etc.*

(We have,
$\theta_{k'}(h) \cdot h' = i$,
$\theta_{k'}(h \cdot s') \cdot h' = \theta_{k'}(s' \cdot h) \cdot h' = \theta_{k'}(s') \cdot \theta_{k'}(h) \cdot h' = u' \cdot i$ and so,
$\theta_{k'}(s') = u'$).

Therefore, edges from two adjacent vertices in a p*-cycle of one induced subgraph is incident on two distinct* p*-cycles in an adjacent induced subgraph over the vertices corresponding to the cosets of* $(\mathbb{Z}_p \times \mathbb{Z}_p)$, *that can be regarded as two adjacent* p*-cycles with respect to the notations used to name the* p*-cycles and vertices (See Remark 2.2.1 for further justifications).*

Remark 2.2.1.

- *If vertices of a* p*-cycle of an induced subgraph over the vertices corresponding to one coset of* $(\mathbb{Z}_p \times \mathbb{Z}_p)$ *is adjacent to vertices along a* p*-cycle of the adjacent subgraph corresponding to another coset of* $(\mathbb{Z}_p \times \mathbb{Z}_p)$, *then*

$h_{m,n} \cdot s \cdot t = h_{m,n} \cdot t \cdot s$, *and so,*
$s \cdot t = t \cdot s$, *which is impossible since* $(\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes \mathbb{Z}_q$ *is a non-abelian group.*

- *Moreover, if* $h_{m_1,n_1} \cdot s \cdot t = h_{m_1,n_1} \cdot t \cdot s^i$*, where* $0 < i < p$ *between the first two adjacent subgraphs considered and* $h_{m_2,n_2} \cdot s \cdot t = h_{m_2,n_2} \cdot t \cdot s^j$*, where* $0 < j < p$ *between the second and third adjacent subgraphs then* $t \cdot s^i = t \cdot s^j$ *and so* $s^i = s^j$ *(*h_{m_1,n_1} *belongs to the first subgraph and* h_{m_2,n_2} *belongs to the second subgraph considered). If the adjacency of vertices between all the adjacent subgraphs corresponding to the cosets of* $(\mathbb{Z}_p \times \mathbb{Z}_p)$ *continue in same manner, the resulting Cayley graph is disconnected which is impossible or if,* $h_{m_1,n_1} \cdot s \cdot t = h_{m_1,n_1} \cdot t \cdot s^i$*, where* $0 < i < p$ *and* $h_{m_2,n_2} \cdot s \cdot t = h_{m_2,n_2} \cdot t \cdot u$*, for some* u $(\neq s)$ *of order* p*, then* $t \cdot s^i = t \cdot u$ *and so* $s^i = u$*, which is again impossible.*

Results and Discussion: I

We first prove Proposition 8, since we need to use it in proving the Theorem 9.

Proposition 8.

Let $X = Cay((\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes \mathbb{Z}_q, S)$*, where* $S = \{s, t\}$*, with* $|s| = p, |t| = q$*, and* $p, q > 2$*. There exists a perfect matching, say* $M1$*, in* $X - v$*, such that* $Y_1 = X - v - M1$ *and* $Y_2 = X - M1 - u$ *are connected and bridgeless, where* $u, v \in V(X)$ *and* v *is adjacent to* u*.*

Proof.

Consider three adjacent induced subgraphs over the vertices corresponding to three cosets of $(\mathbb{Z}_p \times \mathbb{Z}_p)$, say, A, B and C in X, where B is adjacent to A and C as sketched in Figure 3. A, C are adjacent to two other subgraphs, say D and E respectively connected along a q-cycle (or if $q = 3$, then A and C are adjacent to each other).

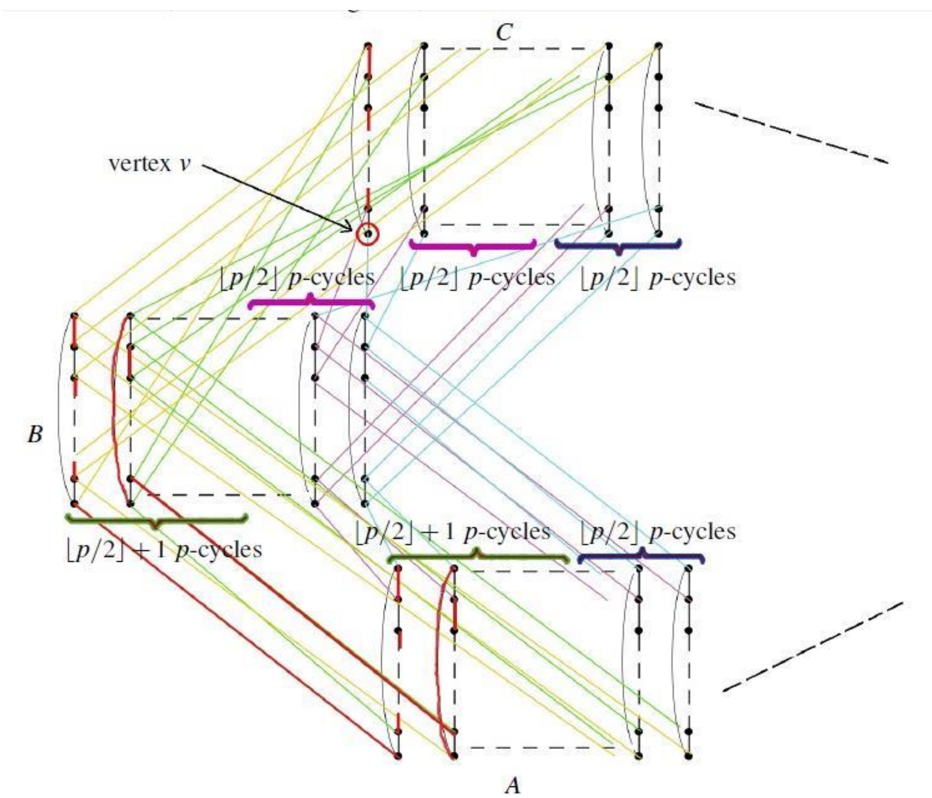

Figure 3: Sketch of an example perfect matching in $X - v$ **(shown in red)**

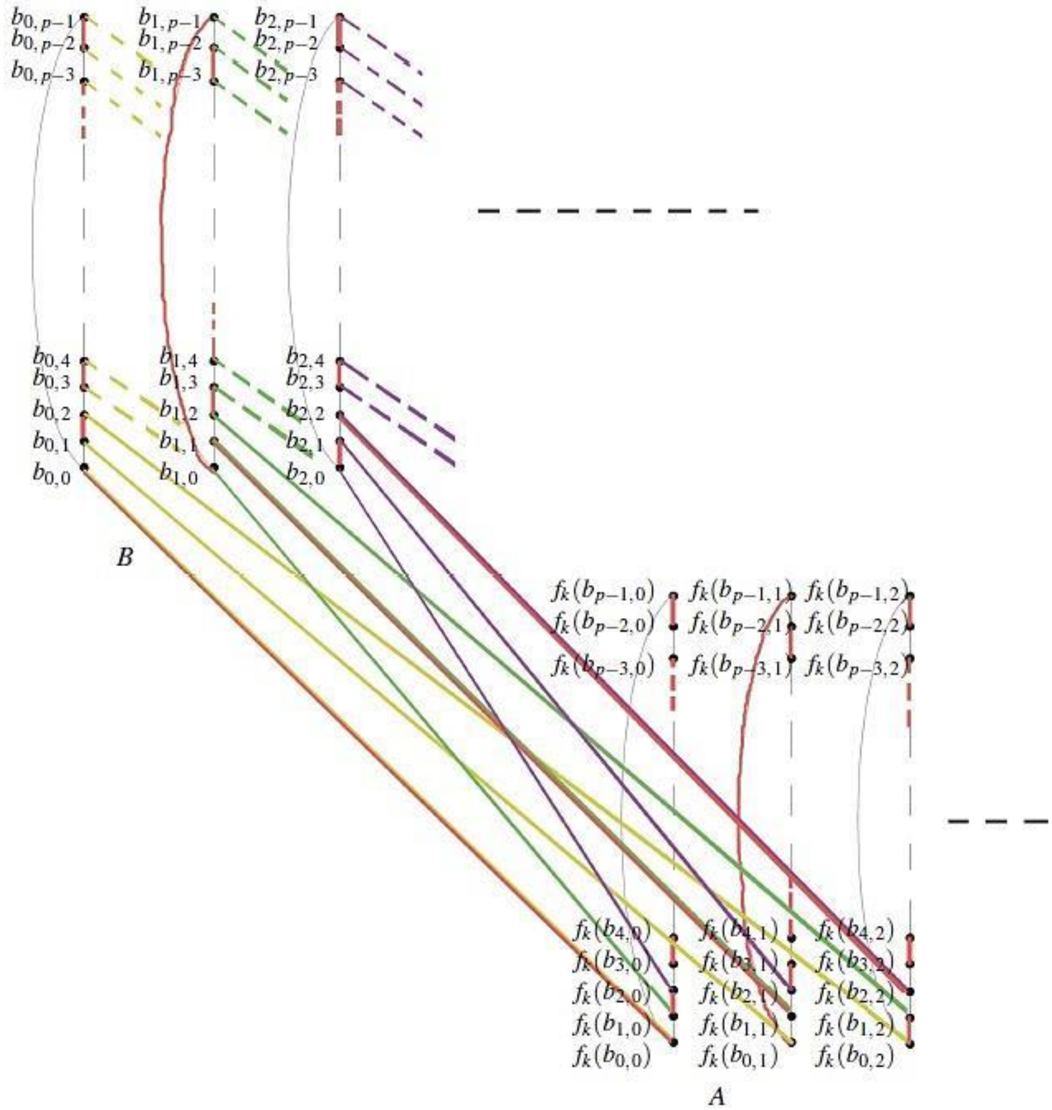

Figure 4: Sketch of edges contributing to the example perfect matching with respect to the subgraphs A and B (shown in red)

We will present an example perfect matching to prove the existence of a perfect matching whose removal results in a connected and bridgeless graph.

Suppose that a vertex v in C is removed when considering a perfect matching in X (it is justifiable to choose any vertex since all the vertices in the Cayley graph are identical). Let, the perfect matching be obtained as described below.

Consider $(\left\lfloor\frac{p}{2}\right\rfloor + 1)$ adjacent p-cycles in B and $(\left\lfloor\frac{p}{2}\right\rfloor + 1)$ adjacent p-cycles in A. Suppose the 0^{th} to $\left\lfloor\frac{p}{2}\right\rfloor^{th}$ p-cycles named with respect to B, A and f_k from B to A are considered (illustrated in Figure 4).

Let's identify the edges contributing to the suggested perfect matching. When considering the edges for the perfect matching, an edge connecting a vertex of a p-cycle in one subgraph to a vertex of a p-cycle in an adjacent subgraph is first chosen, and then the remaining vertices in each p-cycle are paired along the p-cycle.

As an example, consider 0^{th} p-cycles of B and A. Then, let, $b_{0,0} - f_k(b_{0,0})$ be an edge contributing to the perfect matching. Then the edges contributing to the perfect matching obtained by pairing the remaining vertices of each p-cycle are,

$$b_{0,1} - b_{0,2}, \quad b_{0,3} - b_{0,4}, \quad \cdots, \quad b_{0,p-2} - b_{0,p-1},$$
$$f_k(b_{0,0}) \cdot s - f_k(b_{0,0}) \cdot s^2, \quad f_k(b_{0,0}) \cdot s^3 - f_k(b_{0,0}) \cdot s^4, \quad \cdots, f_k(b_{0,0}) \cdot s^{p-2} - f_k(b_{0,0}) \cdot s^{p-1}$$

When considering the 1^{st} p-cycles of B and A, $b_{1,1} - f_k(b_{1,1})$ is an edge contributing to the perfect matching. The edges contributing to the perfect matching, by pairing the vertices along each p-cycle are,

$$b_{1,2} - b_{1,3}, \quad b_{1,4} - b_{1,5}, \quad \cdots, \quad b_{1,p-1} - b_{1,0},$$
$$f_k(b_{1,1}) \cdot s - f_k(b_{1,1}) \cdot s^2, \quad f_k(b_{1,1}) \cdot s^3 - f_k(b_{1,1}) \cdot s^4, \quad \cdots, f_k(b_{1,1}) \cdot s^{p-2} - f_k(b_{1,1}) \cdot s^{p-1}$$

Similarly, when considering the $\left\lfloor\frac{p}{2}\right\rfloor^{th}$ p-cycle of B and A,
$b_{\lfloor\frac{p}{2}\rfloor,\lfloor\frac{p}{2}\rfloor} - f_k(b_{\lfloor\frac{p}{2}\rfloor,\lfloor\frac{p}{2}\rfloor})$ is an edge contributing to the perfect matching and the edges contributing to the perfect matching due to pairing along each p-cycle are,

$$b_{\lfloor\frac{p}{2}\rfloor,\lfloor\frac{p}{2}\rfloor+1} - b_{\lfloor\frac{p}{2}\rfloor,\lfloor\frac{p}{2}\rfloor+2}, \quad b_{\lfloor\frac{p}{2}\rfloor,\lfloor\frac{p}{2}\rfloor+3} - b_{\lfloor\frac{p}{2}\rfloor,\lfloor\frac{p}{2}\rfloor+4}, \quad \cdots, \quad b_{\lfloor\frac{p}{2}\rfloor,\lfloor\frac{p}{2}\rfloor-2} - b_{\lfloor\frac{p}{2}\rfloor,\lfloor\frac{p}{2}\rfloor-1},$$
$$f_k\left(b_{\lfloor\frac{p}{2}\rfloor,\lfloor\frac{p}{2}\rfloor}\right) \cdot s - f_k(b_{\lfloor\frac{p}{2}\rfloor,\lfloor\frac{p}{2}\rfloor}) \cdot s^2, \quad f_k\left(b_{\lfloor\frac{p}{2}\rfloor,\lfloor\frac{p}{2}\rfloor}\right) \cdot s^3 - f_k(b_{\lfloor\frac{p}{2}\rfloor,\lfloor\frac{p}{2}\rfloor}) \cdot s^4, \quad \cdots,$$
$$f_k\left(b_{\lfloor\frac{p}{2}\rfloor,\lfloor\frac{p}{2}\rfloor}\right) \cdot s^{p-2} - f_k(b_{\lfloor\frac{p}{2}\rfloor,\lfloor\frac{p}{2}\rfloor}) \cdot s^{p-1}$$

Let vertex $v = c_{i,j}$, named with respect to C, B and f_{k-1}. In the i^{th} p-cycle of C, pair the remaining vertices and identify the edges contributing to the perfect matching:

$$c_{i,j+1} - c_{i,j+2}, \; c_{i,j+3} - c_{i,j+4}, \quad \cdots, \quad c_{i,j-2} - c_{i,j-1},$$

Next consider $\left\lfloor\frac{p}{2}\right\rfloor$ adjacent p-cycles (adjacent to i^{th} p-cycle as well), say $(i+1)^{th}$ to $(i+\left\lfloor\frac{p}{2}\right\rfloor)^{th}$ p-cycles in C and the remaining $\left\lfloor\frac{p}{2}\right\rfloor$ p-cycles, $(\left\lfloor\frac{p}{2}\right\rfloor+1)^{th}$ to $(p-1)^{th}$ p-cycles in B.

Following the similar pattern considered in identifying the edges contributing to the perfect matching for the 0^{th} to $\left\lfloor\frac{p}{2}\right\rfloor^{th}$ p-cycles of B and A, identify the edges contributing to the perfect matching of $(i+1)^{th}$ to $(i+\left\lfloor\frac{p}{2}\right\rfloor)^{th}$ p-cycles of C and $(\left\lfloor\frac{p}{2}\right\rfloor+1)^{th}$ to $(p-1)^{th}$ p-cycles of B, by considering the edges connecting a vertex of a p-cycle in C and a p-cycle in B and afterwards pairing the remaining vertices of each p-cycle, along the p-cycle for each pair of consecutive p-cycles from C and B.

Similarly, identify the edges contributing to the perfect matching between the remaining $\left\lfloor\frac{p}{2}\right\rfloor$ adjacent p-cycles in C and $\left\lfloor\frac{p}{2}\right\rfloor$ adjacent p-cycles in D and also between the remaining $\left\lfloor\frac{p}{2}\right\rfloor$ adjacent p-cycles in A and $\left\lfloor\frac{p}{2}\right\rfloor$ adjacent p-cycles in E (if q=3 , the remaining $\left\lfloor\frac{p}{2}\right\rfloor$ p-cycles in C and A can be considered).

If D and E are adjacent to each other, consider the remaining $(\left\lfloor\frac{p}{2}\right\rfloor+1)$ adjacent p-cycles in D and E to identify the edges contributing to the perfect matching. Otherwise, there exists two other subgraphs adjacent to each D and E, and consider $(\left\lfloor\frac{p}{2}\right\rfloor+1)$ adjacent p-cycles from each of these subgraphs with the $(\left\lfloor\frac{p}{2}\right\rfloor+1)$ adjacent p-cycles of D and E to identify the perfect matching. Next, if those two subgraphs are adjacent to each other consider the remaining adjacent $\left\lfloor\frac{p}{2}\right\rfloor$ p-cycles to locate the edges contributing to the perfect matching. Else, there exists another two subgraphs adjacent to each of them, and the identification of the edges contributing to the perfect matching can be continued in a similar pattern.

Observe the edges connecting the vertices in the adjacent subgraphs A and B after the removal of the edges contributing to the perfect matching. When considering the 0^{th} p-cycles,

$$b_{0,0} - b_{0,1}, \quad b_{0,2} - b_{0,3}, \quad \cdots, \quad b_{0,p-1} - b_{0,0},$$
$$b_{0,1} - f_k(b_{0,1}), \quad b_{0,2} - f_k(b_{0,2}), \quad \cdots, \quad b_{0,p-1} - f_k(b_{0,p-1}),$$
$$f_k(b_{0,0}) - f_k(b_{0,0}) \cdot s, \quad f_k(b_{0,0}) \cdot s^2 - f_k(b_{0,0}) \cdot s^3, \quad \cdots, f_k(b_{0,0}) \cdot s^{p-1} - f_k(b_{0,0})$$

Considering the 1^{st} p-cycles, the edges remaining are,

$$b_{1,1} - b_{1,2}, \quad b_{1,3} - b_{1,4}, \quad \cdots, \quad b_{1,0} - b_{1,1},$$
$$b_{1,2} - f_k(b_{1,2}), \quad b_{1,3} - f_k(b_{1,3}), \quad \cdots, \quad b_{1,0} - f_k(b_{1,0}),$$
$$f_k(b_{1,1}) - f_k(b_{1,1}) \cdot s, \quad f_k(b_{1,1}) \cdot s^2 - f_k(b_{1,1}) \cdot s^3, \quad \cdots, f_k(b_{1,1}) \cdot s^{p-1} - f_k(b_{1,1})$$

The edges remaining for the other p-cycles can be identified following a similar pattern. Therefore, after the removal of the perfect matching the adjacent subgraphs remain connected.

Since all the adjacent subgraphs remain connected, the graph $Y_1 = X - v - M1$, is connected and there exist two internally disjoint paths between any two vertices in the graph. Therefore by Theorem 2, Y_1 is bridgeless.

Consider the vertex $v = c_{i,j}$. It is connected to the vertices, $c_{i,j-1}, c_{i,j+1}, f_{k-1}(c_{i,j})$ and d_{m_1,n_1} for a vertex $d_{m_1,n_1} \in D$, such that $f_{k-2}(d_{m_1,n_1}) = c_{i,j}$.

Suppose, either $c_{i,j-1}$ or $c_{i,j+1}$ is chosen as u. Let, u be $c_{i,j-1}$ (similar pattern follows for $c_{i,j+1}$). When including v to Y_1 except the edge uv and removing u, the edges $c_{i,j} - c_{i,j+1}, c_{i,j} - f_{k-1}(c_{i,j}), d_{m_1,n_1} - c_{i,j}$ are added to the graph and only the edges $c_{i,j-2} - c_{i,j-1}, c_{i,j-1} - f_{k-1}(c_{i,j-1}), d_{m_2,n_2} - c_{i,j-1}$, where $f_{k-2}(d_{m_2,n_2}) = c_{i,j-1}$ and $d_{m_2,n_2} \in D$, are removed from the graph. Therefore the adjacent subgraphs B, C, and C, D remain connected and hence the graph Y_2 remain bridgeless as well.

Suppose, either $f_{k-1}(c_{i,j})$ or d_{m_1,n_1} was chosen as u. Let, u be $f_{k-1}(c_{i,j})$ (similar pattern follows for d_{m_1,n_1}). When including v to $X - v - M1$ except the edge uv and removing u, the edges $c_{i,j} - c_{i,j+1}, c_{i,j-1} - c_{i,j}, d_{m_1,n_1} - c_{i,j}$ are added to the graph and only the edges $f_{k-1}(c_{i,j}) - f_{k-1}(c_{i,j}) \cdot s, f_{k-1}(c_{i,j}) - f_{k-1}(c_{i,j}) \cdot s^{-1}$ and $f_{k-1}(c_{i,j}) - f_k(f_{k-1}(c_{i,j}))$ are removed from the graph.

Therefore the adjacent subgraphs B, C, and B, A remain connected and hence the graph Y_2 remain connected and bridgeless as well.

■

We have presented a perfect matching by choosing edges according to a pattern through the selection of adjacent p-cycles in adjacent subgraphs, but one can find numerous other perfect matchings such that the removal of the edges of the perfect matching results in a connected, bridgeless graph.

Theorem 9.

The Cayley graph of $(\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes \mathbb{Z}_q$, *with respect to the generating set* $S = \{s, t\}$, *where* $|s| = p, |t| = q$, *with* $p, q > 3$, p *and* q *are distinct primes, is Hamiltonian.*

Proof.

Let $p, q > 3$ (the existence of a Hamilton cycle when $p, q = 2$ or 3 results from Theorem 1).

Consider the construction of a Hamilton cycle in the Cayley graph X as given below.

Let u, v be adjacent vertices in X. If X has cycle decompositions including cycle/s of even length, choose u, v to be on an even cycle (See Remark 3.0.2). There exists a perfect matching in $X - v$. Let it be called $M1$ (The perfect matching $M1$ can be taken in a way such that Y_1 remains connected and bridgeless, by Proposition 8).

From Proposition 8, Y_2 is a connected bridgeless graph with $p^2q - 3$ vertices of degree 3 and 2 vertices of degree 2. Let the two vertices of degree 2 be called w_1 and w_2 and note that w_1 and w_2 are adjacent to u but not adjacent to each other (since $p, q > 3$). Let the edges incident to w_1 be e_1 and $e_1{}'$ and the edges incident to w_2 be e_2 and $e_2{}'$. Join w_1 and w_2 by an edge, say e.

Now Y_2 is a cubic, brigeless, connected graph. Therefore, by Theorem 3, there exists a perfect matching containing any specific edge in X'. Consider a perfect matching in X' including e_1 or $e_1{}'$ or e_2 or $e_2{}'$ so that e will not be included to it. Let, this perfect matching be called $M2$.

Remark 3.0.1. $M1$ *and* $M2$ *are edge disjoint perfect matchings.*

$M1$ spans all the vertices of X except v, and $M2$ spans all the vertices of X except u. The union of the two edge disjoint perfect matchings can either form,

i. a spanning path in the Cayley graph where u and v are the end points of the path or
ii. a path (of odd length) spanning the vertices of the cycle of even length containing u and v with end points at u and v, AND a cycle or a cycle decomposition spanning the remaining vertices of the graph.

Assume case ii. above. Considering the cycle or the cycle decomposition spanning the remaining vertices of the graph, the cycle should be of odd length OR the cycle decomposition should contain at least one cycle of odd length since X has odd order.

Then the odd cycle has to be covered by edges from the union of two edge disjoint perfect matchings $M1$ and $M2$, which is impossible.

Moreover, the number of vertices remaining in the even cycle after the removal of u or the removal of v, is also odd. Checking whether the union of the two perfect matchings will separate the edges along the even cycle forming an odd length path, resulting in a disconnected graph: when considering $M1$ as well as when considering $M2$ by taking the edges belonging to the even cycle, one vertex will remain without getting paired by an edge belonging to the even cycle when considering edges for either perfect matching. Thus, the presence of a disjoint graph as mentioned in ii. is impossible.

Therefore, the union of $M1$ and $M2$ forms a spanning path that has the end points u and v, which is a Hamilton path in X. Since u, v are adjacent join u and v to form a Hamilton cycle in X.

■

Remark 3.0.2.

- *If the two vertices u and v belong to a cycle of odd length as well, will the union of two perfect matchings form a path spanning the vertices of the odd cycle while having a cycle of even length or a cycle decomposition spanning the remaining vertices of the graph?*

Since the vertices belong to the even cycle, it is possible to adjust the edges of the perfect matchings to include edges of the even cycle to result in the same case ii. considered in the above proof. Then the presence of a spanning disjoint graph is impossible since an odd cycle can not be covered by the union of two edge disjoint perfect matchings.

- *It is also possible to think that the union of the two perfect matchings may result in a decomposition consisting of cycles of odd length or a decomposition to even and odd cycles. But in this proof since we consider a union of two edge disjoint perfect matchings, we can conclude that such decompositions are impossible due to the presence of cycle/s of odd length which can not be covered by the edges of the union of two perfect matchings.*

Therefore it is sufficient to consider u, v belonging to an even cycle (whenever even cycles are present).

Results and Discussion: II

We now prove the existence of Hamilton cycles in the Cayley graphs of some semidirect products of finite groups with respect to standard generating sets and the Hamiltonian property of Cayley graphs of groups of orders $p^n q$, $p^2 q^2$ and $p^2 qr$.

Theorem 10.

Let $X_1 = Cay(H, S_1)$ be Hamilton-connected and $X_2 = Cay(H, S_2)$ be Hamiltonian. Then $X = Cay(H \rtimes K, S)$, where $S_1 \subseteq S$ and $S_2 \subseteq S$, is Hamiltonian.

Proof.

The vertices of the Cayley graph X can be arranged such that copies of X_1 and X_2 can be distinctly identifiable. Then, copies of X_1 connected along a cycle resembling a Hamilton cycle in X_2 can be identified.

Let the subgraphs which are copies of X_1 along this cycle be named as $Y_1, Y_2, \cdots, Y_{|K|}$, where $|K| + 1 = 1$, starting at any copy of X_1. The copies of X_1 represents the cosets of H in the Cayley graph X.

Starting at any vertex, say u in Y_1, traverse along a Hamilton path in Y_1 and move to Y_2. In Y_2 again traverse along a Hamilton path and move to Y_3 and continue traversing the vertices of each $Y_4, Y_5, \cdots, Y_{|K-1|}$ in similar manner. Then the path has traversed through $|K - 1||H|$ vertices.

After entering the subgraph $Y_{|K|}$, first identify the vertex v which is connected to vertex u in Y_1. Traverse the vertices of $Y_{|K|}$ along a Hamilton path ending at vertex v, which is possible due to the Hamilton-connectedness of X_1. Finally join v and u which results in a Hamiltonian cycle in X.

■

Remark 4.0.1.

The path traversing the vertices of $Y_1, Y_2, \cdots, Y_{|K-1|}$ has to be adjusted such that it does not enter the subgraph $Y_{|K|}$ at the vertex v. Such adjustment is possible since X_1 is Hamilton-connected.

The following Corollary is a direct consequence of the above Theorem.

Corollary 1.

Let $X_1'' = Cay(\mathbb{Z}_p{}^n, S_1'')$ with $S_1'' = \{s_1, s_2, \cdots, s_n\}$, $|s_1| = |s_2| = \cdots = |s_n| = p$ and $X_2'' = Cay(\mathbb{Z}_q, S_2'')$ with $S_2'' = \{t\}$, $|t| = q$.

Then, $X'' = Cay(\mathbb{Z}_p{}^n \rtimes \mathbb{Z}_q, S''), n \in \mathbb{Z}^+, n \geq 2$ and $p > 2$ where $S_1'' \subseteq S''$ and $S_2'' \subseteq S''$ (S'' is a standard generating set) is Hamiltonian.

Proof.

X_1'', is not a bipartite graph by Theorem 4 (there exists odd cycles, as an example, a Hamiltonian cycle in X_1'' is a p^n-cycle which has odd length since $p > 2$) and $val(X_1'') > 2$. Therefore, by Theorem 5, it is a Hamilton-connected graph and we know that X_2'' is Hamiltonian. Hence X'' is Hamiltonian by Theorem 10.

■

Next we prove that the Cayley graphs of $((\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes \mathbb{Z}_p) \rtimes \mathbb{Z}_q$ with respect to a standard generating set is Hamiltonian.

Theorem 11.

Let,

- $M = Cay(\mathbb{Z}_p \times \mathbb{Z}_p, U_1)$, where $U_1 = \{a, b\}$ with $|a| = |b| = p$,
- $N = Cay(\mathbb{Z}_p, U_2)$, where $U_2 = \{c\}$ with $|c| = p$,
- $X_1 = Cay((\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes \mathbb{Z}_p, S_1)$, where $U_1 \subseteq S_1$ and $U_2 \subseteq S_1$,
- $X_2 = Cay(\mathbb{Z}_q, S_2)$, where $S_2 = \{d\}$ with $|d| = q$ and
- $X = Cay(((\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes \mathbb{Z}_p) \rtimes \mathbb{Z}_q, S)$, where $S_1 \subseteq S$ and $S_2 \subseteq S$.

Then X is Hamiltonian.

Proof.

X_2 is a Hamiltonian Cayley graph and by following the same argument in Corollary 1, X_1 is also Hamiltonian.

Let $p, q > 3$ (the existence of a Hamiltonian cycle when $p, q = 2$ or 3 results from Theorem 1).

The vertices of X can be arranged such that copies of X_1 connected along a q-cycle are distinctly identifiable. The copies of X_1 are the subgraphs representing the cosets of $(\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes \mathbb{Z}_p$ in X.

Let the consecutive copies of X_1 along the q-cycle in clockwise or anti-clockwise direction, be named as $Y_1, Y_2, \cdots, Y_q$ starting with any copy of X_1 and the vertices representing the coset representatives of the cosets of $(\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes \mathbb{Z}_p$, in $Y_1, Y_2, \cdots, Y_q$ be $e_1{}^1, e_2{}^1, \cdots, e_q{}^1$ respectively.

The vertices of X_1 (and so that of copies of X_1) can be arranged such that copies of the Cayley graph M, connected along a p-cycle can be distinctly identifiable. In a subgraph Y_i ($i = 1, \cdots, p$) which is a copy of X_1, let the copies of M along the p-cycle in clockwise or anti-clockwise direction, be named as ${M_1}^i, {M_2}^i, \cdots, {M_p}^i$ starting at the copy of M containing ${e_1}^1$.

Furthermore, since each ${M_j}^i$ ($j = 1, \cdots, p$) are copies of M and Y_i 's are copies of X_1, vertices corresponding to cosets of $(\mathbb{Z}_p \times \mathbb{Z}_p)$ in each copy of X_1 can be identified. Let the vertices corresponding to coset representatives of the cosets of $(\mathbb{Z}_p \times \mathbb{Z}_p)$, in ${M_1}^i, {M_2}^i, \cdots, {M_p}^i$ be ${e_i}^1, {e_i}^2, \cdots, {e_i}^p$.

Starting at ${e_1}^1$ move along a Hamilton path in Y_1 and from the vertex where the path ended move to Y_2 and again move along a Hamilton path. Next, from the vertex where the path ended move to Y_3 and traverse the vertices of Y_3 moving along a Hamilton path and continue to traverse the vertices of each subgraph $Y_4, Y_5, \cdots, Y_{q-1}$ along Hamilton paths in the same manner. Then the path has traversed through $p^3(q-1)$ vertices.

When moving from Y_{q-1} to Y_q if the path is incident on,

i. a vertex belonging to either ${M_2}^q$ or ${M_p}^q$ which are adjacent to ${M_1}^q$ which contains ${e_q}^1$: Suppose the path was incident on a vertex belonging to ${M_2}^q$ (similar construction follows when ${M_p}^q$ is considered). Move along Hamilton paths in each ${M_2}^q, {M_3}^q, \cdots, {M_p}^q$ and after entering ${M_1}^q$ move along a Hamilton path which ends at ${e_q}^1$ which is possible due to the Hamilton-connectedness of M. Also the path traversing the copies of M can be adjusted such that it enters the subgraph ${M_1}^q$ at a vertex other than ${e_q}^1$ due to the same property of M.

ii. a vertex belonging to a copy of M not adjacent to ${M_1}^q$, say ${M_k}^q$: Move along Hamiltonian paths within ${M_k}^q, {M_{k+1}}^q, \cdots, {M_p}^q$ and enter to ${M_1}^q$ at a vertex other than ${e_q}^1$, say u_1. Identify a Hamiltonian path connecting u_1 and ${e_q}^1$. Move along this Hamilton path traversing all vertices except ${e_q}^1$ which is at the end of the path and move to ${M_2}^q$. Suppose the path enters ${M_2}^q$ at a vertex u_2. Identify a Hamilton path from u_2 to ${e_q}^2$ and move along this path traversing all vertices except ${e_q}^2$ and continue to traverse vertices in ${M_3}^q, {M_4}^q, \cdots, {M_{k-2}}^q$ except the vertices ${e_q}^3, {e_q}^4, \cdots, {e_q}^{k-2}$. In ${M_{k-1}}^q$ traverse the path similarly and reach ${e_q}^{k-1}$ as well and traverse back along, ${M_{k-2}}^q, {M_{k-3}}^q, \cdots, {M_1}^q$ traversing ${e_q}^{k-2}, {e_q}^{k-3}, \cdots, {e_q}^2$ and ${e_q}^1$ at the end.

iii. a vertex belonging to ${M_1}^q$, say v_1: Identify a Hamilton path from v_1 to ${e_q}^1$. Move along this Hamilton path traversing all vertices except ${e_q}^1$ which is at the end of the path and move to ${M_2}^q$ or ${M_p}^q$. Suppose the motion to ${M_2}^q$ (similar construction follows for motion to ${M_p}^q$). Then traverse the vertices of each ${M_2}^q, {M_3}^q, \cdots, {M_{p-1}}^q$ moving along Hamilton paths and in ${M_p}^q$ move along a Hamilton path which ends at ${e_q}^p$. From ${e_q}^p$ arrive at the vertex ${e_q}^1$.

The path traversing Y_{q-1} can be adjusted such that it does not enter Y_q at ${e_q}^1$ due to the Hamilton-connectedness of copies of M.

After the constructed path arrive at ${e_q}^1$ it can be connected to ${e_1}^1$ to form a Hamiltonian cycle in X.

■

In Theorem 12, we present a summary of the isomorphism types of groups of orders $p^n q, p^2 q^2$ and $p^2 qr$, where p, q and r are distinct primes for which there exists a Hamiltonian Cayley graph with respect to standard generating sets and the isomorphism types for which there exist a Hamiltonian Cayley graph with respect to any generating set.

Theorem 12.

Let G be a finite group and p, q and r be distinct primes.

1. *if the group present up to isomorphism is one of the following non-abelian groups, every connected Cayley graph is Hamiltonian.*
   - *when* $|G| = p^2q$,

   i. $\mathbb{Z}_q \rtimes \mathbb{Z}_{p^2}$,

   ii. $\mathbb{Z}_{pq} \rtimes \mathbb{Z}_p$,

   iii. $\mathbb{Z}_{p^2} \rtimes \mathbb{Z}_q$
   - *whenever* $pq = 6$ *for* $|G| = p^2q^2$ *and when* $pq \neq 6$*, the following,*

   iv. $\mathbb{Z}_{p^2} \rtimes \mathbb{Z}_{q^2}$,

   v. $\mathbb{Z}_{p^2} \rtimes (\mathbb{Z}_q \times \mathbb{Z}_q)$
   - *when* $|G| = p^3q$,

   vi. $\mathbb{Z}_{p^3} \rtimes \mathbb{Z}_q$,

   vii. $\mathbb{Z}_q \rtimes \mathbb{Z}_{p^3}$,

   viii. $\mathbb{Z}_q \rtimes (\mathbb{Z}_{p^2} \times \mathbb{Z}_p)$,

   ix. $\mathbb{Z}_q \rtimes (\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p)$,

   x. $\mathbb{Z}_q \rtimes ((\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes \mathbb{Z}_p)$,

   xi. $\mathbb{Z}_q \rtimes (\mathbb{Z}_{p^2} \rtimes \mathbb{Z}_p)$
   - *when* $|G| = p^2qr$,

   xii. $\mathbb{Z}_r \rtimes \mathbb{Z}_{p^2q}$,

   xiii. $\mathbb{Z}_{p^2} \rtimes \mathbb{Z}_{qr}$

2. *if the group present up to isomorphism is one of the following, the Cayley graph with respect to standard generating sets is Hamiltonian.*
   - *when* $|G| = p^2q$,

   i. $(\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes \mathbb{Z}_q$
   - *when* $|G| = p^2q^2$,

   ii. $(\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes \mathbb{Z}_{q^2}$,

   iii. $(\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes (\mathbb{Z}_q \times \mathbb{Z}_q)$
   - *when* $|G| = p^3q$,

   iv. $((\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes \mathbb{Z}_p) \rtimes \mathbb{Z}_q$
   - *when* $|G| = p^2qr$,

   v. $(\mathbb{Z}_p \times \mathbb{Z}_r) \rtimes (\mathbb{Z}_p \times \mathbb{Z}_q)$,

   vi. $(\mathbb{Z}_q \times \mathbb{Z}_r) \rtimes \mathbb{Z}_{p^2}$,

   vii. $(\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_q) \rtimes \mathbb{Z}_r$,

   viii. $(\mathbb{Z}_{p^2} \times \mathbb{Z}_q) \rtimes \mathbb{Z}_r$,

   ix. $(\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_r) \rtimes \mathbb{Z}_q$,

Proof.

For the proof of this Theorem, the classification of groups of orders p^2q, p^2q^2 and p^2qr as explained in (Burnside, 1911), (Rajkumar & Devi, 2015) and (Hadi et al., 2018) is considered.

- The existence of a Hamiltonian cycle in every connected Cayley graph in the following groups results from the existing literature.

  The groups,

  $\mathbb{Z}_q \rtimes \mathbb{Z}_{p^2}$

  $\mathbb{Z}_{p^2} \rtimes \mathbb{Z}_q$

$\mathbb{Z}_{p^2} \rtimes \mathbb{Z}_{q^2}$
$\mathbb{Z}_{p^2} \rtimes (\mathbb{Z}_q \times \mathbb{Z}_q)$
$\mathbb{Z}_{p^3} \rtimes \mathbb{Z}_q$
$\mathbb{Z}_q \rtimes \mathbb{Z}_{p^3}$
$\mathbb{Z}_q \rtimes (\mathbb{Z}_{p^2} \times \mathbb{Z}_p)$
$\mathbb{Z}_q \rtimes (\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p)$
$\mathbb{Z}_q \rtimes ((\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes \mathbb{Z}_p)$
$\mathbb{Z}_q \rtimes (\mathbb{Z}_{p^2} \rtimes \mathbb{Z}_p)$
$\mathbb{Z}_r \rtimes \mathbb{Z}_{p^2 q}$
$\mathbb{Z}_{p^2} \rtimes \mathbb{Z}_{qr}$,
consists of Hamiltonian Cayley graphs with respect to any generating set according to Theorem 7.

Every connected Cayley graph of $\mathbb{Z}_{pq} \rtimes \mathbb{Z}_p$ is Hamiltonian by Theorem 6 (for $p, q > 2$. The existence of a Hamilton cycle when $p, q = 2$ results from Theorem 1).

When considering groups of order p^2q^2, when $pq = 6$, since the order of the group is 36, every connected Cayley graph is Hamiltonian by Theorem 1.

- The Hamiltonian property of the Cayley graphs with respect to standard generating sets of,
  $(\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes \mathbb{Z}_q$
  $(\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes \mathbb{Z}_{q^2}$
  $(\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes (\mathbb{Z}_q \times \mathbb{Z}_q)$
  $(\mathbb{Z}_p \times \mathbb{Z}_r) \rtimes (\mathbb{Z}_p \times \mathbb{Z}_q)$
  $(\mathbb{Z}_q \times \mathbb{Z}_r) \rtimes \mathbb{Z}_{p^2}$
  $(\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_q) \rtimes \mathbb{Z}_r$
  $(\mathbb{Z}_{p^2} \times \mathbb{Z}_q) \rtimes \mathbb{Z}_r$
  $(\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_r) \rtimes \mathbb{Z}_q$ results from Theorem 10 (for $p, q, r > 2$)
  and that of,
  $((\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes \mathbb{Z}_p) \rtimes \mathbb{Z}_q$ results from Theorem 11 (for $p, q > 3$).

■

The existence of a Hamiltonian cycle in the Cayley graphs of groups such as $(\mathbb{Z}_{11} \times \mathbb{Z}_{11}) \rtimes \mathbb{Z}_5$, $(\mathbb{Z}_{13} \times \mathbb{Z}_{13}) \rtimes \mathbb{Z}_7$, $(\mathbb{Z}_{11} \times \mathbb{Z}_{11}) \rtimes \mathbb{Z}_{25}$ with respect to any generating set does not result from the existing literature. Therefore, our proofs presented above are indeed new contributions to this subject.

Theorem 1 including the results of this paper can be stated as follows.

Theorem 13.

Let G be a finite group and p, q and r be distinct primes.

1. Every connected Cayley graph on G with respect to any generating set has a Hamiltonian cycle if $|G|$ has any of the following forms
   a) kp, where $1 \leq k \leq 47$,
   b) kpq, where $1 \leq k \leq 5$ and $k = 9$,
   c) pqr,
   d) kp^2, where $1 \leq k \leq 4$,
   e) p^2q, where $p > 5, q > 3$ except for the isomorphism type $(\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes \mathbb{Z}_q$,
   f) p^2q^2, where $p, q > 3$, except for the isomorphism types $(\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes \mathbb{Z}_{q^2}$ and $(\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes (\mathbb{Z}_q \times \mathbb{Z}_q)$

g) kp^3, where $1 \leq k \leq 2$.
h) p^3q, where $p > 3, q > 2$ if the group present up to isomorphism is, $\mathbb{Z}_{p^3} \rtimes \mathbb{Z}_q$ or $\mathbb{Z}_q \rtimes \mathbb{Z}_{p^3}$ or $\mathbb{Z}_q \rtimes (\mathbb{Z}_{p^2} \times \mathbb{Z}_p)$ or $\mathbb{Z}_q \rtimes (\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p)$ or $\mathbb{Z}_q \rtimes ((\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes \mathbb{Z}_p)$ or $\mathbb{Z}_q \rtimes (\mathbb{Z}_{p^2} \rtimes \mathbb{Z}_p)$
i) p^2qr, where $p > 3, q, r > 2$ if the group present up to isomorphism is $\mathbb{Z}_r \rtimes \mathbb{Z}_{p^2q}$ or $\mathbb{Z}_{p^2} \rtimes \mathbb{Z}_{qr}$

2. Cayley graphs with respect to standard generating sets has a Hamiltonian cycle when the $|G|$ has any of the following forms
   a) p^nq, where $n \in \mathbb{Z}^+$, $n \geq 2$ and $p > 2$, when the group present up to isomorphism is ${\mathbb{Z}_p}^n \rtimes \mathbb{Z}_q$,
   b) p^2q^2, where $p, q > 3$, when the groups present up to isomorphism is $(\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes \mathbb{Z}_{q^2}$ or $(\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes (\mathbb{Z}_q \times \mathbb{Z}_q)$,
   c) p^3q, where $p, q > 3$ when the group present up to isomorphism is $((\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes \mathbb{Z}_p) \rtimes \mathbb{Z}_q$,
   d) p^2qr, where $p > 3$, and $q, r > 2$ when the group present up to isomorphism is $(\mathbb{Z}_p \times \mathbb{Z}_r) \rtimes (\mathbb{Z}_p \times \mathbb{Z}_q)$ or $(\mathbb{Z}_q \times \mathbb{Z}_r) \rtimes \mathbb{Z}_{p^2}$ or $(\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_q) \rtimes \mathbb{Z}_r$ or $(\mathbb{Z}_{p^2} \times \mathbb{Z}_q) \rtimes \mathbb{Z}_r$ or $(\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_r) \rtimes \mathbb{Z}_q$.
3. Cayley graph of $(\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes \mathbb{Z}_q$ with respect to a generating set $S = \{s, t\}$, where $|s| = p, |t| = q$, with $p, q > 3$ is Hamiltonian.

Conclusion

In this paper, the existence of a Hamiltonian cycle in several Cayley graphs whose orders have few prime factors is proved. The properties apparent in the Cayley graphs drawn with respect to certain generating sets were analyzed and used to present explicit constructions of Hamiltonian cycles in our proofs. Such analysis is also to be of use for future scholars pursuing studies on this topic.

The future studies of this research are focused on proving the existence of a Hamiltonian cycle in Cayley graphs with respect to other possible types of generating sets for the group orders considered in this paper and also new group orders which will be further contributions to the Theorem 13.

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