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POWER COMPARISON OF THREE ALTERNATIVE STRUCTURAL STABILITY TEST PROCEDURES

by

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1. Introduction

The Author (1977, 1983) considered the problem of equality between sets of coefficients in two linear regressions when disturbance variances are unequal and proposed two small sample test statistics employing two different approahes. Asymptotic theory may also provide a basis to obtain a valid large sampie test procedure to test the above hypothesis.

The purpose of the present paper is to compare the power of these three alternative testing procedures using a Monte Carlo study. Before we move on to the main subject first we consider the three test statistics briefly.

2. The Result

Consider the model

$$y_1 = X_1 \beta_1 + e_1 , \qquad e_1 \sim N (0 , \sigma_1^2 I_{n1}) \qquad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

$$y_2 = X_2 \beta_2 + e_2 , \qquad e_2 \sim N (0 , \sigma_2^2 I_{n2}), \quad E (e_1 e^1_2) = O.$$

Where y_i and X_i are $T_i \times l$ and $T_i \times k$ observations matrices with X_i having full column rank, β_i is a $kX1$ coefficent matrix and e_i is T_iX1 disturbance matrix, for $i = 1, 2$.

Test 1

For $i = 1, 2$, let $\hat{\beta}_i = (X^1_i X_i)^{-1} X_i^1 y_i$ and $e_i = M_i y_i$ and let $M_i = I_{n_i} - X_i (X_i^1 X_i) X^1_i = Z_i Z^1_i$ where $Z^1_i X_i = 0$ and $Z^1_i Z_i = I_{n_i-k}$. It may 1 be noted that the columns of Z_i are eigenvectors of M_i corresponding to unit roots. Then:

$$d = \hat{\beta}_1 - \hat{\beta}_2 \sim N (\nu, \Sigma) \qquad (2)$$

where $\nu = \beta_1 - \beta_2$ and $\Sigma = \sigma_1^2 (X_1^1 X_1)^{-1} + \sigma_2^2 (X_2^1 X_2)^{-1}$.

Now Consider $\overset{*}{e}_i = Z_i^1 \hat{e}_i$ and note that

$$e_i^\star \sim N\ (o,\ \sigma_i^2\ I_{n_i-k}),\quad i = 1,2. \qquad (3)$$

Let r be the largest integer less than, or equal to min($(n_1 - k)/k$, $(n_2 - k)/k$) and partition each $e_i^\star$ into r subvectors $e_i^\star(_1)$.$e^\star_i(_2)$ ......,$e^\star_i(_r)$, each subrector having k elements. Now, let Q_i be a $k \times k$ matrix much that $Q^1_i\ Q_i = (X_i^1\ X_i)_n^{-1}$ Then

$$\eta_j = Q_1^1\ e_1^\star(_j) + Q_2^1\ e_2^\star(_j)\ ,\ j = 1, 2, \ldots, r, \qquad (4)$$

are mutually independent vectors, each distributed as $N\ (O, \Sigma)$ and distributed independently of d.

Finally from T. W. Anderson (1, p. 106) we have that

$$\frac{(\hat\beta_1 - \hat\beta_2)^1 S^{-1} (\hat\beta_1 - \hat\beta_2)}{r}\ \frac{r - k+1}{k} \qquad (5)$$

is distributed as non-central F with k and $r-k+1$ degrees of freedom and non centrality parameter $\nu^1\ \Sigma^{-1}\ \nu$, provided that $r \geqslant k$, where

$$S = \frac{1}{r} \sum_{j=1}^{r} \eta_j\ \eta_j^1 \qquad (6)$$

under the null hypothesis $\beta_1 = \beta_2$ the statistic is distributed as central F. We may therefore use it as a small sample test of equality between sets of coefficients in two linear regressions.

Test 2

Let the model for the t th observation be

$$Y_{1\cdot t} = X_{1\cdot t}\,\beta_1 + e_{1\cdot t}\ ,\quad t = 1, 2, \ldots, n_1 \qquad (7)$$

$$Y_{2\cdot t} = X_{2\cdot t}\,\beta_2 + e_{2\cdot t}\ ,\quad t = 1, 2, \ldots, n_2 \qquad (8)$$

Without loss of generality we can assume the number of observations in the first sample, to be greater than the number of observations in the second sample,, ie. $n_2 < n_1$.

Define $$V_t = Y_{2\cdot t} - \left(\frac{n_2}{n_1}\right)^{\frac{1}{2}} Y_{1\cdot t} + \frac{1}{(n_1 n_2)^{\frac{1}{2}}} \sum_{i=1}^{n_2} Y_{1\cdot i} - \bar{Y}_1;$$

$$t = 1, 2, \ldots, n_2 \qquad (9)$$

where $\bar{Y}_1$ is the mean value of the observations in Y_1 .

Substituting for $Y_{1,t}$ and $Y_{2,t}$ in (9) and from (7) and (8) we have

$$V_t = X_{2,t}\beta_2 + e_{2,t} - \left(\frac{n_2}{n_1}\right)^{\frac{1}{2}} [X_{1,t}\beta_1 + e_{1,t}]$$

$$+ \frac{1}{(n_1 n_2)^{\frac{1}{2}}} \sum_{i=t}^{n_2} [X_{1,i}\beta_1 + e_{1,i}] - [\bar{X}_1\beta_1 + \bar{e}_1] \quad — \quad (10)$$

where $\bar{X}_1$ is the row vector of columns means of X_1 matrix. Rearranging the terms in (10) we have,

$$V_t = P_t\,\beta_2 + Q_t\,\beta_1 + U_t \quad — \quad (11)$$

where $P_t = X_2$; $Q_t = -\left(\frac{n_2}{n_1}\right)^{\frac{1}{2}} X_{1,t} + \frac{1}{(n_1 n_2)^{\frac{1}{2}}} \sum_{i=1}^{n_2} X_{1,i} - \bar{X}_1$,

and $U_t = e_{2,t} - \left(\frac{n_2}{n_1}\right)^{\frac{1}{2}} e_{1,t} + \frac{1}{(n_1 n_2)^{\frac{1}{2}}} \sum_{i=1}^{n_2} e_{1,i} - \bar{e}_1$

$t = 1, 2, \ldots\ldots, n_2.$

For this transformed model (11) we have,

$$E\,Ut = O \;;\ \text{for } t = 1,2,\ldots\ldots, n_2$$

$$E\,U_t^2 = \sigma_2^2 + \left(\frac{n_2}{n_.}\right)\sigma_1^2 \;;\quad \text{for } i = 1,2,\ldots, n_2$$

and $E\,U_t\,U_s = O.$ for $t \neq s$; $t, s = 1,2,\ldots\ldots n_2.$

Therefore the transformed model (11) satisfies the conditions of the classical linear model.

Collecting all the observations we can write the model (11) as follows:

$$V = P\,\beta_2 + \theta\,\beta_1 + U \quad — \quad (12)$$

Where V is the $n_2 \times 1$ vector of observations on the transformed dependent variable, and P and Q are $n_2 \times k$ matrices of observations on the transformed independent variables, β_1 and β_2 are $k \times 1$ coefficient vectors and V is the $n_2 \times 1$ vector of transformed disturbances.

Write $V = ZB. + U$ — (13)

where $Z = [P\ Q]$ and $B = \begin{bmatrix}\beta_2 \\ \beta_1\end{bmatrix}$

To test the hypothesis $\beta_1 = \beta_2$ in (13) now we can use the conventional F test and it follows that under the null hypothesis $\beta_1 = \beta_2$ the statistic

$$\frac{(\hat{\beta}_1 - \hat{\beta}_2)^1 \quad [A_{11} - A_{21} - A_{12} + A_{22}]^{-1} \quad (\hat{\beta}_1 - \hat{\beta}_2) / k}{V^1 V \quad (n_2 - 2k)} \tag{14}$$

is distributed on F distribution with k and $(n2-{}_2k)$ degrees of freedom, where $\hat{\beta}_1$ and $\hat{\beta}_2$ are the OLS estimators of β_1 and β_2, V is the OLS residuals from the model (12) and A's are given by the following expression:

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} P^1P & Q^1Q \\ Q^1P & \theta^1Q \end{bmatrix}$$

Test 3

Considering the model (1) we have

$$\hat{\beta}_1 - \hat{\beta}_2 \sim N\ (v, \Sigma)$$

under the null hypothesis $\beta_1 - \beta_2$

$$(\hat{\beta}_1 - \hat{\beta}_2) \sim (O, \Sigma).$$

Therefore, using the distribution theory we have

$$(\hat{\beta}_1 - \hat{\beta}_2)^1 \ \Sigma^{-1} \ (\beta_1 - \beta_2) \sim \chi^2_k$$

or

$$(\hat{\beta}_1 - \hat{\beta}_2)^1 \ [\sigma_1^2 \ (X_1{}^1 X_1)^{-1} + \sigma_2^2 \ (X_2{}^1 X_2)^{-1}] \ (\hat{\beta}_1 - \hat{\beta}_2) \sim \chi^2_{k},$$

If we replace σ_1^2 and σ_2^2 by $\hat{\sigma}_1^2$ and $\hat{\sigma}_2^2$ where $\hat{\sigma}_1^2 = \hat{e}_1{}^1 \ \hat{e}_1 \ / \ n_1 - k$ and $\hat{\sigma}_2^2 = \hat{e}_2^1 \ \hat{e}_2 \ / \ n_2 - k$, then under the null hypothesis of $\beta_1 = \beta_2$ the statistic

$$(\hat{\beta}_1 - \hat{\beta}_2)^1 \left[\hat{\sigma}_1^2 \ (X_1^1 X_1)^{-1} + \hat{\sigma}_2^2 (X_2^1 X_2)^{-1}\right]^{-1} (\hat{\beta}_1 - \hat{\beta}_2)$$

is distributed of asymptically as χ^2 distribution with k degrees of freedom.

3. A Monte Carlo Study

In the above section we introduced three alternative testing procedures to test the equality of coefficients in two linear regressions when the disturbance variances are unequal. Two of them (Test 1 and Test 2) are small sample tests while the other test (Test 3) is valid only for large samples. In this section we will compare the power of these testing procedures using the Monte carlo technique.

We generate pseudo — random numbers from a normal distribution with zero mean and some specified standard deviation for the disturbances and generate the values for independent variables from a uniform distribution. Then we obtain the values for the dependent variable using the relation $y = \chi \beta + e$ by giving certain values to the coefficient vector β.

Computation of the test statistics

First we consider the statistics 1. To construct this statistics we need to calculate sets of recursive residuals. For this purpose we employ Farebrothe's (1975) "INCLUDE 2" procedure. This procedure is a revised version of Gentleman's (1974) "INCLUDE" procedure which use $\bar{\theta} R$ decomposition of the weighted data matrix

$$W_i^{\frac{1}{2}} \left[X_i \quad Y_i \right] = \bar{\theta} j \; D_i^{\frac{1}{2}} \left[\bar{R}_i \quad \bar{\theta}_i \right]$$

to evaluate

$$\frac{w^{\frac{1}{2}}_{j+1} \left(\gamma_{j+1} - x_{j+1} \; bj \right)}{\left[1 + w_{j+1} - x_{j+2} \left(X^{1}_{j} \; W_{j} \; X_{j} \right)^{-1} x_{j+1} {}^{\frac{1}{2}} \right]}$$

recursive residuals. Since we are only interested in unweighted recursive residuals in this study, we can proceed by putting $w_j = 1$ for all j.

Since $X = Q \; D^{\frac{1}{2}} \; R$

$$(X^1 X) = \bar{R}^1 \; D^{\frac{1}{2}} \; \bar{Q}^1 \; Q \; D^{\frac{1}{2}} \; R = R^1 \; D^{\frac{1}{2}} \; D^{\frac{1}{2}} \; R$$

where $(X^1 \; X)^{-1} = R^{-1} \; D^{-\frac{1}{2}} \; D^{-\frac{1}{2}} \; (\bar{R})^{-1} = Q^1 \; Q$

where $Q = D^{-\frac{1}{2}} \quad (R^1)^{-1}$

Therefore we use the same procedure to obtain Q matrix in the statistic 1. In the complete program we employ a procedure named QGET to obtain the above matrix product. The procedure REGRESS used to get the estimates of the parameter vector β.

Once we have obtained these quantities, *ie.* the recussive residuals, the Q matrix and $\hat{\beta}$, the calculation of the test Statistic 1 is straight forward Also the calculation of the test Statistics 2 and 3 is not very difficult because all the required quantities are obtained from the above procedures.

We conduct several experiments each based on 1000 realisations. The experiments differ according to (1) the size of the samples; (2) the number of independent variables in the models, and (3) the X matrices of the models

Case 1

In this section we consider all three test procedures together and compare the power performance in various combinations of the size of the two samples. Here we exclude the intercept terms from the models to avoid the multi. collinearity problem in Test 2. We used the following two models in our experiments:

(1) $Y_1 = 2.5\ X_1 + 3.0\ X_2 + e_1,\ ;\ e_1 \sim N\ (0,1)$
$Y_1 = 2.7\ X_3 + 3.2\ X_4 + e_2\ ;\ e_2 \sim N\ (0,5)$

(2) $Y_1 = 10.0 - 2.0\ X_2 + e_1\ ;\ e_1\ N \sim (0,1)$
$Y_2 = 10.0\ \ 0\ X_3 - 1.7\ X_4 + e_2\ ;\ e_2 \sim N\ (0,5)$

To generate X matrices we used two uniform distributions with lower and upper limits (−1, 1) and (10,50) and conducted several experiments varying the combination of the size of the two samples. The empirical significance levels and the powers of the tests were calculated by taking the nominal level of significance as 0.05. The results are presented in Tables 1 and 2 where the quantities are defined as follows:

n_1 = Size of the first sample

n_2 = Size of the second sample

α_1 = empirical significance level of the Test 1

α_2 = empirical significance level of the Test 2

α_3 = empirical significance level of the Test 3

P_1 = empirical power of the Test 1

P_2 = empirical power of the Test 2

P_3 = empirical power of the Test 3

Table 1; Model No. (1)

n_1 n_2	α_1	α_2	α_3	P_1	P_2	P_3
6, 6	0.049	0.057	0.118	0.280	0.556	1.000
10, 6	0.054	0.057	0.165	0.229	0.434	0.995
20, 6	0.049	0.060	0.145	0.283	0.656	1.000
30, 6	0.052	0.052	0.139	0.277	0.703	1.000
20, 12	0.054	0.054	0.108	1.000	1.000	1.000

Table 2: Model No. (2)

n_1 n_2	α_1	α_2	α_3	P_1	P_2	P_3
6, 6	0.049	0.057	0.118	0.411	0.847	1.000
10, 6	0.054	0.057	0.165	0.333	0.701	1.000
20, 6	0.049	0.060	0.145	0.411	0.903	1.000
30, 6	0.052	0.052	0.139	0.405	0.945	1.000
20, 12	0.054	0.054	0.108	1.000	1.000	1.000

These results show that the power of test 2 is always greater than that of test 1 but has less power compared to Test 3. There is no definite pattern of variation of power of test 1 when increasing the size of one sample alone. But except one case (n_1, n_2 = 10, 6) the power of test 2 increases when increasing the size of one sample. Both test 1 and 2 keep their respective significance levels very close to the nominal level but test 3 does not. When we increase the size of both samples the power of test 1 and test 2 will improve and obtain the maximum when $n_1 = 20$, $n_2 = 12$.

Table 3: Model No. (1)

n_1 n_2	α_1	α_2	α_3	P_1	P_2	P_3
6, 6	0.050	0.057	0.111	0.504	0.941	1.000
10, 6	0.057	0.055	0.165	0.395	0.856	1.000
20, 6	0.050	0.061	0.146	0.528	0.978	1.000
30, 6	0.047	0.053	0.142	0.491	0.990	1.000
20, 12	0.050	0.052	0.109	1.000	1.000	1.000

Table 4: Model No. (2)

n_1 n_2	α_1	α_2	α_3	P_1	P_2	P_3
6, 6	0.050	0.057	0.111	0.696	0.998	1.000
10, 6	0.057	0.055	0.165	0.558	0.983	1.000
20, 6	0.050	0.061	0.146	0.718	1.000	1.000
30, 6	0.047	0.053	0.142	0.687	1.000	1.005
20, 12	0.050	0.52	0.109	1.000	1.000	1.000

Tables 3 and 4 show the variation of power and significance level of three tests when we used the same models but different X matrices. The matrices were generated by using the two uniform distributions with lower and upper limits (–2,2) and (100, By Comparing the results of the corresponding tables we can see that the significance levels remain unchanged in both cases but the power of the tests has increased in the second case. In both cases the pattern of variation of the power of tests 1 and 2 remains the same.

Case 2

Here we compare the power performance of the three test procedures when the model contain three independent variables. As in the case 1 we exclude the intercept terms from the models. We used the following models for our experiment:

$$Y_1 = 10.0\ X_1 - 3.0\ X_2 + 0.5X_3 + e_1\ ;\ e_1 \sim N\ (0,1)$$
$$Y_2 = 10.2\ X_4 - 3.2\ X_5 + 0.2X_6 + e_2\ ;\ e_2 \sim N\ (0,5)$$

We generated the X matrices using three uniform distributions with lower and upper limits (–1, 1), (10, 50) and (5, 10). The results are presented in Table 5. As in the case 1 the power of the test 2 is always greater than that of the test 1 and there is no definite pattern of variation of the power of test 1. test 3 is always superior to both test 1 and test 2.

Table 5

n_1,n_2	α_1	α_2	α_3	P_1	P_2	P_3
12, 12	0.054	0.049	0.078	0.373	1.000	1.000
20, 12	0.053	0.050	0.095	0.415	1.000	1.000
30, 12	0.54	0.045	0.089	0.480	1.000	1.000
40, 12	0.053	0.046	0.117	0.410	1.000	1.000
20, 20	0.053	0.054	0.087	1.000	1.000	1.000

Table 6

n_1,n_2	α_1	α_2	α_3	P_1	P_2	P_3
12, 12	0.054	0.049	0.078	0.349	0.999	1.000
20, 12	0.053	0.050	0.095	0.343	0.999	1.000
30, 12	0.054	0.045	0.089	0.421	1.000	1.000
40, 12	0.053	0.046	0.117	0.357	1.000	1.000
20, 20	0.053	0.054	0.887	1.000	1.000	1.000

In Tables 5 & 6 we presented the results of the experiments conducted using the same model but different X matrices. Here we generated the X matrices using three uniform distributions with lower and upper limits (– 2,2), (10,50) and (3,12). By looking at tables 5 and 6 we can see that although the new X matrix reduces the power of test 1 the pattern remains unaltered. In both cases the power of Test 1 is always less than that of test 2 which agreed with the results in case 1. test 2 and 3 have maximum power in all situations but test 1 has maximum power only when both samples are considerably large. (*ie* $n_1 = n_2 = 20$).

4. Concluding Remarks

According to this Monte Carlo study we can draw the following conclusions. The power of the asymptotic test, even in small samples, is always greater than that of the other small sample tests, but it fails to keep the significance level very close to the nominal level. Most of the cases the small sample tests achieve the exact value of the significance level, as suggested by the theory. With respect to the two small sample tests we can conclude that the power of test 2 is always greater than that of test 1. However, both these tests have considerable power when the coefficients of the models differ markedly. In most cases they attain the maximum power when the size of the two samples is roughly 20.

REFERENCES

1. FAREBROTHER, R. W. "Recursive Residuals: A Remark on Algohlithm AS75: Basic procedures for Large, sparse or weighted Least Squares Problem *Applied Statistics* (1976)

2. GENTLEMAN, W. M. "Algorithm AS 75; Basic procedures for Large, sparse or weighted Least squares Problems,, *Applied Statistics* (1974)

3. JAYATISSA, W. A. "Tests of equality between sets of coffiecients in two linear regressing when disturbance variances are unequal, *Econometrica Vol* 45, *No.* 5 (1977)

4. JAYATIS;A, W. A. "Adoptation of Scheffe's procedure o test of equality between sets of coefficients in two linear regressions when disturbance variancy aie unequal,,, *Vidyodaya J., Arts, Sci. let Vol.* II, *No.* 1, & 2 (1933)

5. SMITH V K. *Monte Carlo Methods*; D C. Heath and Co , Lexington. Messachusetts (1973)